LECTURE NOTES ON

The Star Complement Technique

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1 Introduction

Star sets and star partitions were first introduced by Cvetković, Rowlinson and Simić in 1993 as a way to study eigenspaces of graphs and also to investigate the graph isomorphism problem [40]. Simultaneously, Ellingham [41] introduced the notion of star complement (he used the name μ-basis). Seemingly, the word star complement was first used by Rowlinson in [32]. In 1995, the star partitions and the related notion of star bases were used in [35] to reestablish a result of Babai *et al.* [42] which states that the graph isomorphism can be done in polynomial time for graphs with bounded eigenvalue multiplicities. Apparently, there has not been done much work after this result on star partitions. On the other hand, star sets and star complements have been proved to be helpful ideas and a lot of researches have been devoted to these topics. Mainly, they have been used to give bounds on the size of graphs and also to give characterizations of some well know graphs.

Let H be a graph of order t with no eigenvalue μ. The star complement technique is a method to construct a graph G with H as an induced subgraph and having μ as an eigenvalue of multiplicity equal to $n-t$ (n, the order of G). H is said to be a star complement for μ in G. The technique has been used for example to determine all exceptional graphs. Another main result asserts that if $\mu \neq -1, 0$, then $n \leq \binom{t+1}{2}$ and therefore there are a finite number of graphs G. The method has also been used to characterize some well known families of graphs by their star complements. The technique seems promising and hopefully more problems related to graph spectra will be resolved in the future by this method.

2 Orthogonal projection

We will need the notion of orthogonal projection to present an algebraic definition of star sets. Let W be a vector subspace of $\mathbb{R}^n$. Let $\mathbf{x}$ be a vector. The *projection* of $\mathbf{x}$ on W is defined as a vector $y \in W$ such that $x-y$ is orthogonal to any vector in W, i.e. $x-y \in W^\perp$ (the inner product is the standard one). It is easily seen that the projection exists and is unique. If $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ is an orthonormal basis for W, then the projection of $\mathbf{x}$ on W is given by $\sum_{i=1}^{m} < \mathbf{x}, \mathbf{u}_i > \mathbf{u}_i$. Note that the projection of $\mathbf{x}$ on $\mathbb{R}^n$ is itself and so if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is an orthonormal basis for $\mathbb{R}^n$, we have $\mathbf{x} = \sum_{i=1}^{n} < \mathbf{x}, \mathbf{v}_i > \mathbf{v}_i$.

Let W and V be vector subspaces of $\mathbb{R}^n$. Then the *orthogonal projection* of V onto W is the linear map from V into W which sends any $x \in V$ to its projection on W. Note that the kernel of this map is $V \cap W^\perp$. Let $V = \mathbb{R}^n$. Then the n by n matrix P representing the orthogonal projection is a matrix whose columns are the projections of the standard basis of $\mathbb{R}^n$ represented also in the standard basis. Note that the columns of P span W. Let S be a matrix whose columns form a basis for W. Then it is easy to see $P = S(S^TS)^{-1}S^T$. Note that $P^2 = P^T = P$.

The *angle* between two vectors $\mathbf{x}$ and $\mathbf{y}$ is defined as $< \mathbf{x}, \mathbf{y} > /(|\mathbf{x}||\mathbf{y}|)$. The *angle* between $\mathbf{x}$ and W is defined as the angle between $\mathbf{x}$ and its projection on W.

3 Star sets and star complements

It is well known that a symmetric matrix of rank r has a full rank principle submatrix of order r. A proof is given in the following lemma.

Lemma 1 *Let* $M = \begin{bmatrix} A & B^T \\ B & C \end{bmatrix}$, *where* A *and* C *are symmetric and the columns corresponding to* A *constitute a basis for* M. *Then* A *is full rank.*

Proof Every column of B^T is a linear combination of columns of A which means that every row of B is a linear combination of rows of A. $\square$

Let A be a real symmetric matrix whose columns and rows are indexed by $\{1, 2, \ldots, n\}$ and let μ be its eigenvalue. Let P be the matrix representing the orthogonal projection of $\mathbb{R}^n$ onto the eigenspace $\mathcal{E}(\mu)$. Since the columns of P span $\mathcal{E}(\mu)$, we can choose a set of linearly independent columns of P to be a basis for $\mathcal{E}(\mu)$. Using this we present an algebraic definition of star sets. Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ denote the standard orthonormal basis of $\mathbb{R}^n$.

Definition A subset X of $\{1, 2, \ldots, n\}$ is called a *star set* if the vectors $P\mathbf{e}_j (j \in X)$ form a basis for $\mathcal{E}(\mu)$.

We also have a combinatorial definition of star sets.

Definition A subset X of $\{1, 2, \ldots, n\}$ is called a *star set* if the matrix obtained from A by removing rows and columns corresponding to X does not have μ as an eigenvalue. In graph theory context, a *star set* for an eigenvalue μ in G is a subset X of vertices such that μ is not an eigenvalue of $G \setminus X$. The graph $G \setminus X$ is called a *star complement* for μ in G.

We now show that the two definitions are equivalent. We give it for graphs, but it also holds for any symmetric matrix.

Theorem 1 *Each of the following is a necessary and sufficient condition for a k-subset X of $V(G)$ to be a star set for the eigenvalue μ of multiplicity k:*

(i) $P\mathbf{e}_j (j \in X)$ *is a basis of* $\mathcal{E}(\mu)$.

(ii) $\mathcal{E}(\mu)$ *has a basis of eigenvectors* $\mathbf{x}_s (s \in X)$ *such that* $\mathbf{x}_s^T \mathbf{e}_t = \delta_{st}$ *whenever* $s, t \in X$.

(iii) μ *is not an eigenvalue of* $G \setminus X$.

(iv) $\mathbb{R}^n = \mathcal{E}(\mu) \oplus \mathcal{V}$, *where* $\mathcal{V} =< \mathbf{e}_j : j \notin X >$.

Proof (i) $\to$ (ii): P has a full rank principle submatrix of order k by Lemma 1. So the corresponding columns of P (which are eigenvectors) can be transformed into the standard form. (ii) $\to$ (iii): Every column of $\mu I - A$ corresponding to X is a linear combination of columns not corresponding to X, so the result follows by Lemma 1. (iii) $\to$ (iv): We have $\mathcal{E}(\mu) \cap \mathcal{V} = 0$. So the assertion follows. (iv) $\to$ (i): Let $P\mathbf{e}_j (j \in X)$ be dependent. Then there is a nonzero vector $\mathbf{x} \in \mathcal{E}(\mu)^\perp$ such that it is zero on $\overline{X}$. We have $\mathbf{x} = \mathbf{y} + \mathbf{z}$, where $\mathbf{y} \in \mathcal{E}(\mu)$ and $\mathbf{z}$ is zero on X. So $\mathbf{x}$ and $\mathbf{y}$ coincide on X and therefore using $\mathbf{xy} = 0$, we have $\mathbf{x} = 0$, a contradiction. □

The existence of star sets can also be seen in other ways. Let G be graph with eigenvalue μ of multiplicity k. Then there exists a vertex v such that $G \setminus v$ has eigenvalue μ of multiplicity $k-1$. This is since the multiplicity of μ in $G \setminus v$ is $k-1, k$ or $k+1$ (by interlacing theorem). Now by

$$P'(G) = \sum_v P(G \setminus v)$$

it is clear that for some v, the multiplicity should be $k-1$. So there exists a set X of vertices of size k such that $G \setminus X$ does not have μ as an eigenvalue. We also have the following reasoning. Since $\mu I - A$ has rank $n-k$, it has a principle submatrix $\mu I - C$ of order $n-k$ such that C has no eigenvalue μ.

Since removing a vertex from a graph changes the multiplicity of an eigenvalue by at most one, we have the following result.

Theorem 2 *Let X be a star set for eigenvalue μ of multiplicity k in G and let $S \subseteq X$. Then μ is an eigenvalue of $G \setminus S$ of multiplicity $k - |S|$.*

Corollary 1 *Let G be a graph with X and H as star set and star complement, respectively. Then for any $Y \subseteq X$, $G \setminus Y$ has H as a star complement.*

The following theorems are also straightforward.

Theorem 3 *Let Y be a subset of $V(G)$ such that $G \setminus Y$ does not have μ as an eigenvalue. Then there is a star set X such that $X \subseteq Y$.*

Theorem 4 *Let S be a subset of $V(G)$ such that $G \backslash S$ has μ as an eigenvalue of multiplicity $k - |S|$ (k, the multiplicity of μ). Then there is a star set X such that $S \subseteq X$.*

4 Star partitions

This section is independent from the subsequent sections and so it may be neglected. Let A be a real symmetric matrix whose columns and rows are indexed by $\{1, 2, \ldots, n\}$. Let $A = \sum_{i=1}^{m} \mu_i P_i$ be the spectral decomposition of A, where μ_i are distinct eigenvalues and P_i denotes the matrix representing the orthogonal projection of $\mathbb{R}^n$ onto the eigenspace $\mathcal{E}(\mu_i)$. Since the columns of P_i span $\mathcal{E}(\mu_i)$, we can choose a set of linearly independent columns of P_i to be a basis for $\mathcal{E}(\mu_i)$. The question is that if we can find these bases such that their corresponding indices of columns partition $\{1, 2, \ldots, n\}$.

Definition A partition $X_1 \cup X_2 \cup \cdots \cup X_m$ of $\{1, 2, \ldots, n\}$ is called a *star partition* if the vectors $P_i \mathbf{e}_j (j \in X_i)$ form a basis for $\mathcal{E}(\mu_i)$ for $i = 1, \ldots, m$. The set of vectors $P_i \mathbf{e}_j (j \in X_i)$ is called a *star basis* for $\mathcal{E}(\mu_i)$ and the set of vectors $P_i \mathbf{e}_j (j \in X_i)$ $(1 \leq i \leq m)$ is called a *star bases* for $\mathbb{R}^n$.

The motivation for investigating star bases is the graph isomorphism problem. This is not to be discussed here.

Theorem 5 *Any real symmetric matrix has a star partition.*

Proof Let $\mu_1, \mu_2, \ldots, \mu_m$ be distinct eigenvalues of real symmetric matrix A of order n. Let $\{R_1, R_2, \ldots, R_m\}$ be a partition of $\{1, 2, \ldots, n\}$ and let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be an orthonormal basis of $\mathbb{R}^n$ such that $\{\mathbf{x}_j : j \in R_i\}$ is a basis of $\mathcal{E}(\mu_i)$. Consider the transition matrix $T = (t_{ij})$ from the basis $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ to the standard basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$. So we have

$$\mathbf{e}_j = \sum_{k=1}^{n} t_{kj} \mathbf{x}_k.$$

Multiplying by P_i, we get

$$P_i\mathbf{e}_j = \sum_{k\in R_i} t_{kj}\mathbf{x}_k.$$

By the multiple Laplacian expansion of $\det(T)$, we have

$$\det(T) = \sum\left\{\pm\prod\det(T_i)\right\},$$

where the summation is taken over all partitions of columns of T of type same as the partition $\{R_1, R_2, \ldots, R_m\}$. Since $\det(T) \neq 0$, for some partition we have that $\prod\det(T_i)$ is nonzero, and so $\det(T_i) \neq 0$ for all i. Now since $\{\mathbf{x}_j : j \in R_i\}$ is a basis of $\mathcal{E}(\mu_i)$, we find that $\{P_i\mathbf{e}_j : j \in R_i\}$ is a basis of $\mathcal{E}(\mu_i)$ for all i and consequently the assertion follows. □

Example 1 A star partition for the Petersen graph is shown in Figure 1. This graph has 750 distinct star partitions which fall into 10 nonisomorphic classes determined by the automorphism of the graph [33].

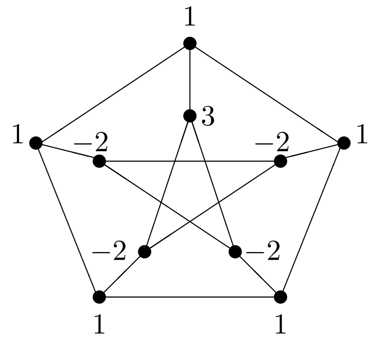

Figure 1: A star partition

For a discussion on star partitions in well known graphs such as complete graphs, complete bipartite graphs, cycles, paths and so on see Chapter 3 of [26].

With a slight modification in the proof of Theorem 5, we can prove a stronger result.

Theorem 6 *Let X be a star set in G. Then there is a star partition consisting of X as one of its elements.*

Note that one cannot always extend two star sets to a star partition. A counterexample is the Petersen graph.

The next theorem follows immediately from Theorem 1.

Theorem 7 *Each of the following is a necessary and sufficient condition for the partition $X_1 \cup X_2 \cup \cdots \cup X_m$ of $V(G)$ to be a star partition:*

(i) For each i, $P_i\mathbf{e}_j (j \in X_i)$ is a basis of $\mathcal{E}(\mu_i)$.

(ii) For each i, $\mathbb{R}^n = \mathcal{E}(\mu_i) \oplus \mathcal{V}_i$, where $\mathcal{V}_i =< \mathbf{e}_j : j \notin X_i >$.

(iii) For each i, μ_i is not an eigenvalue of $G \setminus X_i$.

(iv) For each i, $\mathcal{E}(\mu_i)$ has a basis of eigenvectors $\mathbf{x}_s (s \in X_i)$ such that $\mathbf{x}_s^T \mathbf{e}_t = \delta_{st}$ whenever $s, t \in X_i$.

The following theorem is obvious by Theorem 7(iv).

Theorem 8 *If the eigenspaces of the (labeled) graphs G_1 and G_2 coincide for all pairs μ_{1i} and μ_{2i} $(1 \le i \le m)$, then the star partitions of G_1 are precisely the star partitions of G_2.*

Theorem 9 *Let G be a connected regular graph whose complement is also connected. Then they have the same star partitions.*

Proof Since the adjacency matrices of G and $\overline{G}$ commute, there is a common basis of eigenvectors. Also since both are regular and connected, the multiplicities of the eigenvalue pairs $(\mu, -1-\mu)$ and also of the pair $(\lambda_{\max}, n-1-\lambda_{\max})$ (n, the order of G) are the same and therefore the eigenspaces coincide for all pairs and the assertion follows from Theorem 8. □

Star partitions and star bases were originally introduced as a way to investigate the graph isomorphism problem. For a graph we can associate a specific star bases which we call canonical and it is minimal in some ordering of bases. Then it turns out that two graphs are isomorphic if and only if they have the same eigenvalues and canonical star bases. Finding a star partition (and so a star bases) can be done in polynomial time. If there is a polynomial time algorithm to determine the canonical star bases, then the graph isomorphism problem will be polynomial. See [33] for a comprehensive discussion of the subject.

5 Operations on graphs

The following theorem is trivial. We state it since it is useful in finding star sets for disconnected graphs.

Theorem 10 *Let $G = G_1 + G_2$. Then $X = X_1 \cup X_2$ is a star set for G if and only if X_1 and X_2 are star sets in G_1 and G_2, respectively.*

6 Galaxy graphs

A graph is called *galaxy* if all star sets for all eigenvalues are independence sets. Examples include graphs with all eigenvalues being simple, stars, double stars and book graphs. Some results concerning these graphs are given in [28].

One also may consider the case where all star sets induce complete graphs.

7 Conjugate and opposite eigenvalues

Theorem 11

(*i*) *Let μ_1 and μ_2 be conjugate (i. e. they have the same minimal polynomial over rationals). Then X is a star set for μ_1 if and only if it is a star set for μ_2.*

(*ii*) *Let μ_1 and μ_2 be opposite in a bipartite graph (i. e. $\mu_1 = -\mu_2$). Then X is a star set for μ_1 if and only if it is a star set for μ_2.*

Proof (i) Since conjugate eigenvalues have the same multiplicity in a graph ??????????, the assertion easily follows. (ii) The eigenspace of μ_2 is obtained from the eigenspace of μ_1 by negating entries in each eigenvector corresponding to one part of the vertices. Now apply Theorem 1(ii). □

8 μ-rank

Let μ be an eigenvalue (of multiplicity m) of a graph G of order n. Then $t = n - m$ is called *μ-rank* of G. Note that μ-rank is equal to the eigenspace codimension of the eigenvalue μ. Also note that μ-rank is the rank of $\mu I - A$, A the adjacency matrix of G. Hence 0-rank is precisely the rank of G.

Note that if we add a vertex to a graph or remove a vertex from a graph, then the multiplicity of an eigenvalue changes by at most one. If we add a vertex to a graph, then its μ-rank increases by at most 2. If we remove a vertex from a graph, then its μ-rank decreases by at most 2.

9 Structural considerations for $\mu \neq -1, 0$

Theorem 12 *Let X be a star set and $s \in X$. Then the neighborhood of x in $\overline{X}$ is nonempty or x is isolated and $\mu = 0$.*

Proof Suppose that the neighborhood of x in $\overline{X}$ is empty. Then $H = G - (X \setminus \{x\})$ has μ as an eigenvalue of multiplicity 1. Removing x from H only changes the multiplicity of eigenvalue 0. Hence $\mu = 0$. Now Suppose that x is adjacent to y in X. Consider an eigenvector which is 1 on y and zero on $X \setminus \{y\}$ (Theorem 1(ii)). Then the sum of entries on the the neighborhood of x is 1, a contradiction. □

This theorem is especially useful for cubic graphs since it shows that a star set in a cubic graph is a union of cycles and matching.

The closed neighborhood of a vertex is the set obtained by adding the vertex itself to its neighborhood. The following is also straightforward using Theorem 1(ii).

Theorem 13 *Let X be a star set and $s, t \in X$. If the neighborhoods of s and t in $\overline{X}$ are the same, then*

- *$s \sim t$, $\mu = -1$ and s and t have the same closed neighborhood or*
- *$s \not\sim t$, $\mu = 0$ and s and t have the same neighborhood.*

Corollary 2 *If $\mu \notin \{-1, 0\}$, then the neighborhoods of elements of X in $\overline{X}$ are distinct and nonempty.*

Corollary 3 *For given t, there are only finitely many graphs of μ-rank t, where $\mu \notin \{-1, 0\}$.*

Theorem 14 *Let X be a star set and let the multiplicity of μ be k and k' in G and $G \setminus \overline{X}$, respectively. Then there are at least $k - k'$ vertices in $\overline{X}$ which have a nonempty neighborhood in X.*

Proof For $u \in X$, let $\Gamma(u) = \{v \sim u : v \in X\}$ and $\overline{\Gamma}(u) = \{v \sim u : v \in \overline{X}\}$. Let $\overline{\Gamma}(X) = \bigcup_{u \in X} \overline{\Gamma}(u)$.

Let P denote the matrix representing the orthogonal projection of $\mathbb{R}^n$ onto the eigenspace $\mathcal{E}(\mu)$. From $PA = \mu P$, for $u \in X$ we have

$$\sum_{v \in \overline{\Gamma}(u)} P\mathbf{e}_v = \mu P\mathbf{e}_u - \sum_{v \in \Gamma(u)} P\mathbf{e}_v.$$

We show that these vectors $\mu P\mathbf{e}_u - \sum_{v \in \Gamma(u)} P\mathbf{e}_v$ span a space of dimension $k - k'$ and so $\sum_{v \in \overline{\Gamma}(u)} P\mathbf{e}_v$ is a subspace of dimension $k - k'$ of space spanned by $\{P\mathbf{e}_j : j \in \overline{\Gamma}(X)\}$ which yields $|\overline{\Gamma}(X)| \geq k - k'$. Consider the linear transformation

$$P\mathbf{e}_u \to \mu P\mathbf{e}_u - \sum_{v \in \Gamma(u)} P\mathbf{e}_v.$$

It has the matrix $\mu I - A_X$ (A_X, the adjacency matrix of $G \setminus \overline{X}$) in the basis $P\mathbf{e}_u$ and since it has rank $k - k'$, the assertion follows. □

10 Structural considerations for $\mu = -1, 0$

A graph is called *reduced* if it has no duplicated vertices, i.e. vertices with the same neighborhood. A graph is called *coreduced* if it has no coduplicated vertices, i.e. vertices with the same closed neighborhood. Note that from a given graph, we finally find a unique graph which is reduced (coreduced) by shrinking vertices with the same neighborhood (closed neighborhood) to one vertex step by step (the order does not matter).

Theorem 15 *Let H be a star complement in graph G for $\mu = 0$ and suppose that A, the adjacency matrix of G, is of the form*

$$A = \begin{pmatrix} A_X & B^T \\ B & C \end{pmatrix},$$

where C is the adjacency matrix of H. Let $N = [B \;\; C]$. Then G is reduced if and only if the columns of N are distinct.

Proof If G is reduced, then the result follows from Theorem 13. The converse is trivial. □

Theorem 16 *Let H be a star complement in graph G for $\mu = -1$ and suppose that A, the adjacency matrix of G, is of the form*

$$A = \begin{pmatrix} A_X & B^T \\ B & C \end{pmatrix},$$

where C is the adjacency matrix of H. Let $N = [B \quad C + I]$. Then G is coreduced if and only if the columns of N are distinct and nonzero.

Proof If G is coreduced, then the result follows from Theorems 12 and 13. The converse is trivial. □

Corollary 4 *For given t and $\mu = 0$ ($\mu = -1$), there are only finitely many reduced (coreduced) graphs of μ-rank t.*

11 First bounds

Theorem 17 *Let G be a graph of order n and μ-rank t. Then*

- *If $\mu \neq -1, 0$, then $n \leq 2^t + t - 1$.*
- *If $\mu = 0$ and G is reduced, then $n \leq 2^t$.*
- *If $\mu = -1$ and G is coreduced, then $n \leq 2^t - 1$.*

Proof For $\mu \neq -1, 0$, the result follows from Corollary 2. For $\mu = 0$ and $\mu = -1$ the result follows from Theorems 15 and 16, respectively. □

For non-integral eigenvalues, there is a linear bound given in the following theorem.

Theorem 18 *Let G be a graph of order n and μ-rank t. If μ has l conjugates ($l \geq 2$, if μ is not integral) and the index of G is not equal to μ or any of its conjugates, then*

$$n \leq \frac{t-1}{l-1} + t.$$

Proof Let μ be of multiplicity k. Then any of its conjugates is also of multiplicity k ?????????? and the assertion follows. □

12 Connectivity

In the proof of the following theorem we use the observation that a graph is connected if and only if we can order its vertices such that each vertex (except for the first one) is adjacent to a preceding vertex.

Theorem 19 *A connected graph has a connected star complement for any eigenvalue.*

Proof Let A be the adjacency matrix of a connected graph G. Suppose that G is of μ-rank t. We order the vertices such that each vertex (except for the first one) is adjacent to a preceding vertex. We now choose t columns of $\mu I - A$ starting from the first column and omitting any column which is a linear combination of the preceding columns. By Lemma 1, we obtain an induced subgraph H with no eigenvalue μ. We prove that H is connected by showing that each vertex (except for the first one) is adjacent to a preceding vertex. Let $y > 1$ be a vertex of H and let j be the first vertex adjacent to it in G. Then row y in $\mu I - A$ has -1 in the position (y, j) and has zero in all positions (y, i) for $i < j$. This shows that column j belongs to our selected columns and hence j is a vertex of H. □

As a converse, we have the following.

Theorem 20 *Let H be a star complement for $\mu \neq 0$ in graph G and let X be the star set. Then*

(i) If H is connected, then G is connected.

(ii) If H is 2-connected, then one of the following holds:

- *G is 2-connected.*
- *$\mu \neq -1$ and G has a pendant edge at a vertex of $\overline{X}$.*
- *$\mu = -1$ and G has a cut vertex v in $\overline{X}$ whose neighbors in X induce a complete subgraph which is a component of $G \setminus v$.*

Proof (i) is trivial by Theorem 12.

We prove (ii). Suppose that H is 2-connected while G has a cutvertex v. By Theorem 12, $v \in \overline{X}$. Consider the neighborhood $\Gamma(v)$ of v in X. Note that all vertices $(\overline{X} \setminus \{v\}) \cup$

$(X \setminus \Gamma(v))$ belong to a component $\mathcal{C}$ of $G \setminus v$. Hence the vertices of $\Gamma(v)$ are distributed in some components $\mathcal{C}_i$ of $G \setminus v$ and also probably in $\mathcal{C}$. If $\mu \neq -1$, then by Theorem 13, $\mathcal{C}_i$ have only one vertex and we are done. Now let $\mu = -1$. ?????????????????????????? □

13 Extension by a vertex

Theorem 21 *Let G' be a graph obtained from G by adding a new vertex v and characteristic vector* $\mathbf{b}$. *Therefore, G' has the adjacency matrix*

$$A = \begin{pmatrix} 0 & \mathbf{b}^T \\ \mathbf{b} & A \end{pmatrix},$$

where A is the adjacency matrix of G. Then $m_{G'}(\mu) = m_G(\mu)+1$ if and only if $\mathbf{b} \in \mathcal{E}_G(\mu)^\perp$ and G' has a μ-eigenvector which is nonzero in the new vertex.

Proof Let $m_{G'}(\mu) = m_G(\mu) + 1$. Consider a star set X for μ in G. Then $X \cup \{v\}$ is a star set for μ in G'. By Theorem 1, $\mathcal{E}_{G'}(\mu)$ has a basis of eigenvectors which has an identity matrix form on $X \cup \{v\}$. Therefore, G' has a μ-eigenvector which is nonzero in v and $\mathbf{b} \in \mathcal{E}_G(\mu)^\perp$. The converse is similarly proved using star set and Theorem 1. □

14 Graph dominance

Definition A subset D of $V(G)$ is called a *dominating set* if each vertex in $\overline{D}$ is adjacent to a vertex in D. A dominating set D is a *location-dominating set* if for any distinct pair $u_1, u_2 \in \overline{D}$, we have $\Gamma(u_1) \cap D \neq \Gamma(u_2) \cap D$. The *dominating number* (respectively, *location-dominating number*) of a graph is the least cardinality of a dominating set (location-dominating set). By Theorems 12 and 13, we have the following.

Theorem 22 *Let G be a graph with no isolated vertices. Then any star complement in G is a dominating set and it is also a location-dominating set if $\mu \notin \{-1, 0\}$.*

For the proof of the following theorem, see [38].

Theorem 23 *Let X be a star set for $\mu \notin \{-1, 0\}$ in a regular graph G. Let $\overline{X}$ be a minimal dominating set. Then*

- *If each vertex of $\overline{X}$ is isolated in $G \setminus X$, then $\mu = 1$ and G is a union of K_2.*
- *If no vertex of $\overline{X}$ is isolated in $G \setminus X$, then $\mu = 1$ and G is a union of Petersen graphs.*

For more results on the connections of dominating sets with star sets, see [22].

15 Regular graphs with regular star complement

A star set X is called *uniform* if all vertices in $\overline{X}$ have the same number of neighbors in X. So if G is regular and X is uniform, then the star complement is also regular and we have the following corollary. We denote the neighborhood of a vertex v in a graph G by $G(v)$.

Theorem 24 *Let X be a uniform star set in a graph G, say $|G(v) \cap X| = b$ for all $v \in \overline{X}$. If μ is not a main eigenvalue of G, then $G \setminus \overline{X}$ is regular of degree $\mu + b$.*

Proof For $u \in X$, let $\Gamma(u) = \{v \sim u : v \in X\}$ and $\overline{\Gamma}(u) = \{v \sim u : v \in \overline{X}\}$.

Let P denote the matrix representing the orthogonal projection of $\mathbb{R}^n$ onto the eigenspace $\mathcal{E}(\mu)$. From $PA = \mu P$, for $u \in X$ we have

$$\sum_{v \in \overline{\Gamma}(u)} P\mathbf{e}_v = \mu P\mathbf{e}_u - \sum_{v \in \Gamma(u)} P\mathbf{e}_v.$$

Summing over X , we obtain

$$\sum_{v \in \overline{X}} bP\mathbf{e}_v = \mu \sum_{u \in X} P\mathbf{e}_u - \sum_{u \in X} d_u P\mathbf{e}_u,$$

where d_u denotes the degree of u in $G \setminus \overline{X}$. Since $P\mathbf{j} = 0$, we obtain

$$\sum_{u \in X} (\mu - d_u + b)P\mathbf{e}_u, = 0$$

However, $P\mathbf{e}_u$ are independent and so $d_u = \mu + b$. □

Corollary 5 *Let a r-regular graph H be a star complement for μ in a k-regular graph G and let X be the star set. Then one of the following holds:*

- *$\mu = k, r = k - 1$ and each component of G is a complete graph.*

- $r-k \le \mu \le r-1$ *and* X *induces a regular subgraph of degree* $\mu+k-r$.

For more results and examples and also a discussion on cubic graphs, see [37].

16 Reconstruction

Theorem 25 (The Reconstruction Theorem and its Converse) *Let* X *be a set of* k *vertices in a graph* G *and suppose that* A*, the adjacency matrix of* G*, is of the form*

$$A = \begin{pmatrix} A_X & B^T \\ B & C \end{pmatrix},$$

where A_X *is the adjacency matrix of the subgraph induced by* X*. Then* X *is a star set for* μ *in* G *if and only if* μ *is not an eigenvalue of* C *and*

$$\mu I - A_X = B^T(\mu I - C)^{-1}B.$$

In this situation, $\mathcal{E}(\mu)$ *consists of the vectors of the form*

$$\begin{pmatrix} \mathbf{x} \\ (\mu I - C)^{-1}B\mathbf{x} \end{pmatrix},$$

where $\mathbf{x} \in \mathbb{R}^k$ *and so*

$$\begin{pmatrix} \mathbf{e}_u \\ (\mu I - C)^{-1}\mathbf{b}_u \end{pmatrix}, u \in X$$

is a basis for $\mathcal{E}(\mu)$.

Proof Let X be a star set for μ. By definition, μ is not an eigenvalue of C. We now show that

$$\mu I - A_X = B^T(\mu I - C)^{-1}B.$$

We have

$$\mu I - A = \begin{pmatrix} \mu I - A_X & -B^T \\ -B & \mu I - C \end{pmatrix}.$$

By definition, $\mu I - A$ and $\mu I - C$ are of rank $n-k$. Therefore, there is L such that

$$[\mu I - A_X \quad - B^T] = L[-B \quad \mu I - C].$$

So we have $\mu I - A_X = -LB$ and $-B^T = L(\mu I - C)$ and the assertion follows. For the converse, it is sufficient to show that the vectors $\begin{pmatrix} \mathbf{x} \\ (\mu I - C)^{-1}B\mathbf{x} \end{pmatrix}$, where $\mathbf{x} \in \mathbb{R}^k$ are eigenvectors for μ in G and it is easily proved by a simple calculation. The result then follows from Theorem 3. □

The theorem can also be stated in the following form. It is just a statement of the compatibility graph (see the next section).

Theorem 26 *Let X be a set of k vertices in a graph G and suppose that A, the adjacency matrix of G, is of the form*

$$A = \begin{pmatrix} A_X & B^T \\ B & C \end{pmatrix},$$

where A_X is the adjacency matrix of the subgraph induced by X. Then X is a star set for μ in G if and only if μ is not an eigenvalue of C and $(G \setminus X) + u + v$ has μ as an eigenvalue of multiplicity 2 for every distinct pair $u, v \in X$.

Remark If μ is not an eigenvalue of a matrix C, then $(\mu I - C)^{-1}$ is expressed as a polynomial in C (using the minimal polynomial of C). We have the following.

Lemma 2 *Suppose that μ is not an eigenvalue of a matrix C with the minimal polynomial*

$$m(x) = x^{d+1} + c_d x^d + c_{d-1} x^{d-1} + \cdots + c_1 x + c_0.$$

Then

$$m(\mu)(\mu I - C)^{-1} = a_d C^d + a_{d-1} C^{d-1} + \cdots + a_1 C + a_0 I,$$

where for $0 \leq i \leq d$

$$a_{d-i} = \mu^i + c_d \mu^{i-1} + c_{d-1} \mu^{i-2} + \cdots + c_{d-i+1}.$$

Notation Let $\mathbf{x}$ and $\mathbf{y}$ be two column vectors of dimension equal to the order of $\mu I - C$. Then we define

$$< \mathbf{x}, \mathbf{y} >= \mathbf{x}^T (\mu I - C)^{-1} \mathbf{y}.$$

Remark Let $\mathbf{b}_u$ denote the columns of B. Then we have

$$< \mathbf{b}_u, \mathbf{b}_v >= \begin{cases} \mu & \text{if } u = v, \\ -1 & \text{if } u \sim v, \\ 0 & \text{otehrwise.} \end{cases}$$

Let $S = [B \ \ C - \mu I]$. Then it is easy to see that

$$\mu I - A = S^T (\mu I - C)^{-1} S.$$

This implies

$$< \mathbf{s}_u, \mathbf{s}_v >= \begin{cases} \mu & \text{if } u = v, \\ -1 & \text{if } u \sim v, \\ 0 & \text{otehrwise,} \end{cases}$$

where $\mathbf{s}_i$ denote the columns of S.

17 The compatibility graph

From the reconstruction theorem (Theorems 25 and 26), it is seen that if G has H as a star complement, then any induced subgraph of G containing H also has H as a star complement. Therefore, we need only to consider maximal graphs containing H. For $\mu = 0$ ($\mu = -1$), note that if G is reduced (coreduced), then any induced subgraph of G containing H is also reduced (coreduced) (see Theorems 15 and 16).

Let H be a graph with no eigenvalue μ and $|V(H)| = t$. Let C be the adjacency of H. We define *the compatibility graph* of H and μ as follows: The vertices are (0,1) vectors $\mathbf{b}_u$ of dimension t such that

$$< \mathbf{b}_u, \mathbf{b}_u >= \mu,$$

and two vertices $\mathbf{b}_u$ and $\mathbf{b}_v$ are adjacent if and only if

$$< \mathbf{b}_u, \mathbf{b}_v >= -1, 0.$$

(For $\mu = 0, -1$, we also have that the vectors $-\mathbf{b}$ are not the columns of $\mu I - C$.)

Now the problem of finding maximal graphs (reduced or coreduced in cases $\mu = -1, 0$) having H as a star complement for eigenvalue μ is equivalent to find the maximal cliques in the compatibility graph of H and μ.

Example 2 We find the maximal graphs for the star complement H for eigenvalue 1, where H is the pentagon 123451. From Lemma 2, we have

$$(I - C)^{-1} = 3I - C^2 = \begin{bmatrix} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 & -1 \\ -1 & 0 & 1 & 0 & -1 \\ -1 & -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 \end{bmatrix}.$$

We should have $< \mathbf{b}_u, \mathbf{b}_u >= \mathbf{b}_u^T (I - C)^{-1} \mathbf{b}_u = 1$. We denote $\mathbf{b}_u$ by a set S, where S is the set of corresponding nonzero value vertices in $\mathbf{b}_u$. A simple calculation shows that S consists of a single vertex or three consecutive vertices of the pentagon. Hence, the compatibility graph has 10 vertices and is illustrated in Figure 2. The automorphism group of H has three orbits on maximal cliques (of orders 2,3 and 5). These determine three maximal graphs shown in Figure 3, where the vertices of H are circled.

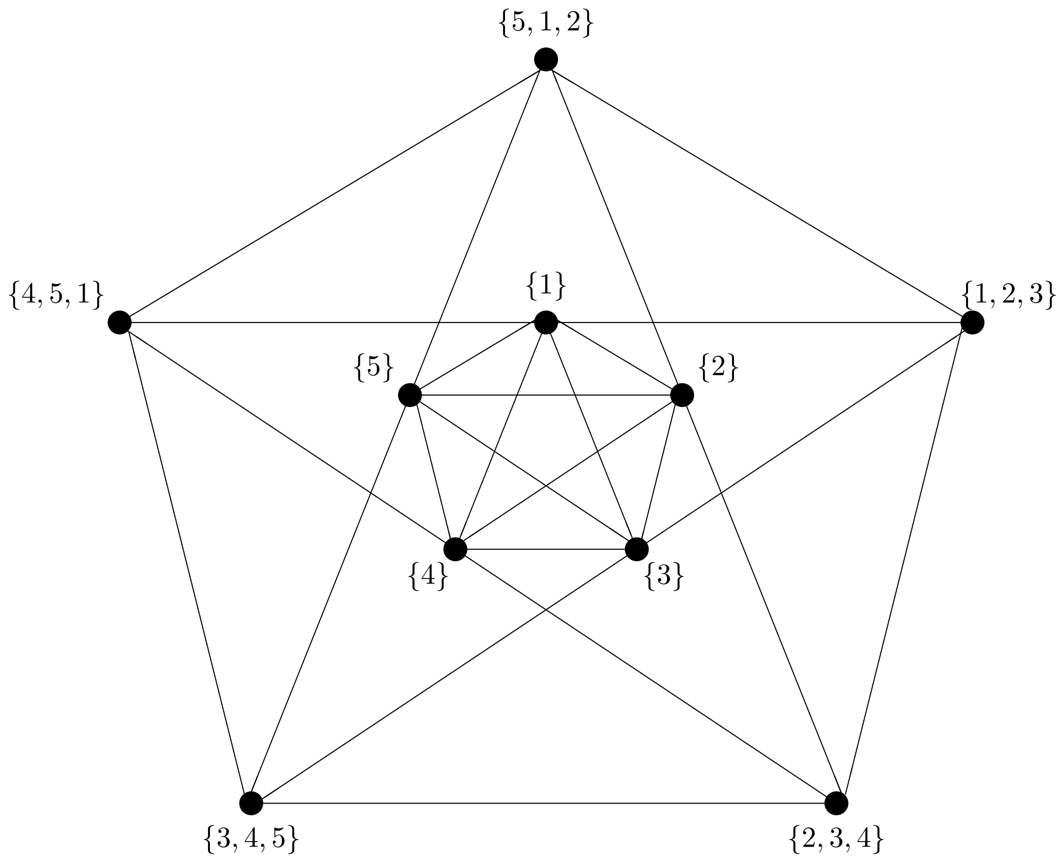

Figure 2: The compatibility graph for the pentagon and eigenvalue 1

18 $\mathcal{G}(\mu, t)$

For given μ and t, we saw that the set $\mathcal{G}(\mu, t)$ of graphs (reduced or coreduced in cases $\mu = 0, -1$) of μ-rank t is finite (Corollaries 3 and 4). Consider the poset $\mathcal{G}(\mu, t)$ with the

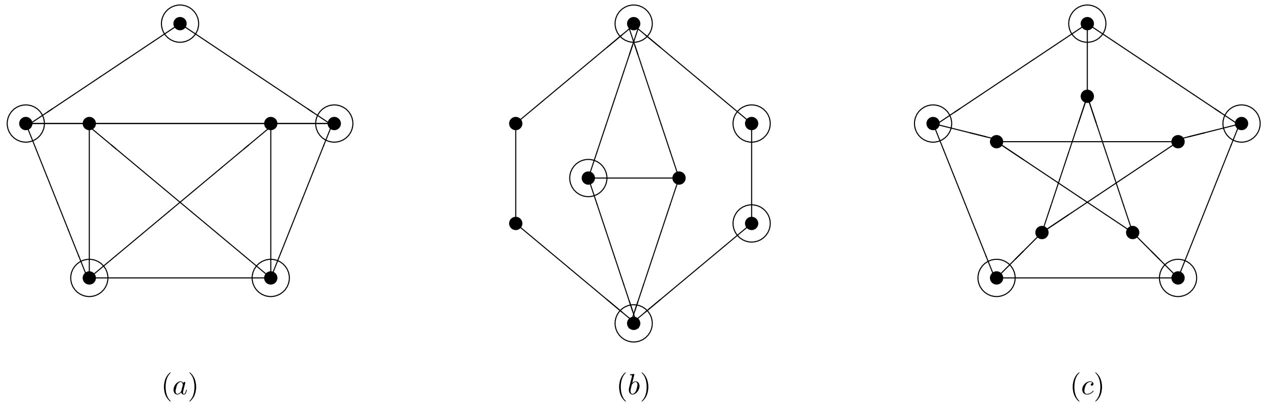

Figure 3: The maximal graphs with C_5 as a star complement for 1

induced subgraph partial order. The minimal elements of this poset are precisely graphs of order t with no eigenvalue μ. Any element has at least one induced subgraph of order t which is a star complement for eigenvalue μ. It is easy to see that the maximal elements of this poset are exactly those maximal graphs obtained from the compatibility graphs of all graphs of order t with no eigenvalue μ. The compatibility graphs provide an algorithm to generate $\mathcal{G}(\mu, t)$. Note that if G in $\mathcal{G}(\mu, t)$, it is not true that any induced subgraph of G is also in $\mathcal{G}(\mu, t)$.

19 Maximal elements of $\mathcal{G}(0, t)$

First we give some elementary properties of maximal elements of $\mathcal{G}(0, t)$. Note that such elements always have an isolated vertex. However, we consider these elements without their isolated vertices. In the following, $G(u)$ dentes the neighborhood of u.

Lemma 3 *Let G be a maximal element of $\mathcal{G}(0, t)$. Let u and v be two nonadjacent vertices such $G(u) \cap G(v) = \varnothing$. Then there is a vertex w such that $G(w) = G(u) \cup G(v)$.*

Proof Trivial. □

This lemma shows that such elements are connected and non-bipartite (except for K_2). We also have the following.

Lemma 4 *Every maximal element of $\mathcal{G}(0,t)$ has diameter at most* 3.

Proof Consider two vertices u and v. Then apply Lemma 3 to v and some $x \in G(u)$. Then the assertion easily follows. □

It is shown that K_n is a maximal and at the same time a minimal element of $\mathcal{G}(0,n)$ [41]. Some other families of graphs which are maximal in $\mathcal{G}(0,t)$ are also known (see [1, 41]). Also all reduced graphs with rank at most 7 are found in [1]. For rank 8, all maximum reduced graphs are obtained [1].

20 $m(\mu,t)$

The maximum order of a graph in $\mathcal{G}(\mu,t)$ is denoted by $m(\mu,t)$. In [1], it is conjectured that

$$m(0,t) = \begin{cases} 2^{(t+2)/2} - 1 & \text{if } t \text{ is even,} \\ 5.2^{(t-3)/2} - 1 & \text{if } t \text{ is odd and } t > 1. \end{cases}$$

In [1, 34], by a simple construction it is shown that the above provides a lower bound for $m(0,t)$. An upper bound is given in Theorem 17. For $\mu \neq -1, 0$, we will give a quadratic bound in the next section. However, for $\mu = 0$, it is not possible to give a polynomial bound since there are examples of exponential orders. Here is an example: Consider a reduced graph G of order n and rank t. Duplicate each vertex and add a new vertex and connect it to one of pairs of vertices. The resulting graph G' is of order $2n+1$ and rank $t+2$. It has the adjacency matrix

$$\begin{bmatrix} A & A & \mathbf{j}^t \\ A & A & 0 \\ \mathbf{j} & 0 & 0 \end{bmatrix},$$

where A is the adjacency matrix of G. Starting form a single vertex we found a graph of order $2^{(t+2)/2} - 1$ and rank t for any even t. Kotlov and Lovasz [34] have shown that for a reduced graph of rank t, the number of vertices is less than $c2^{t/2}$ for some constant c. One question which arises is that what happens for coreduced graphs. Note that if we drop the property of being reduced (coreduced), then the order of graph is not bounded from the above.

21 $m(-1,t)$

We conjecture that

$$m(-1,t) = \begin{cases} 2^{(t+2)/2} - 2 & \text{if } t \text{ is even,} \\ 5.2^{(t-3)/2} - 2 & \text{if } t \text{ is odd and } t > 3. \end{cases}$$

We give a construction which shows that the above provides a lower bound for $m(-1,t)$. Consider a coreduced $(n/2-1)$-regular graph G of order n and (-1)-rank t. Duplicate each vertex and add two new vertices which are not joined together and connect each one to one of pairs of vertices. The resulting graph G' is an n-regular graph of order $2n+2$ and (-1)-rank $t+2$. $G'+I$ is of the form

$$\begin{bmatrix} A+I & A+I & \mathbf{j}^t & 0 \\ A+I & A+I & 0 & \mathbf{j}^t \\ \mathbf{j} & 0 & 1 & 0 \\ 0 & \mathbf{j} & 0 & 1 \end{bmatrix},$$

where A is the adjacency matrix of G. Starting from $2K_1$ for even (-1)-rank and cube for odd (-1)-rank we found families of graphs which attain the bound above. For $t \leq 9$, we have checked that the conjecture is true and there exists a unique graph attaining the bound above.

22 Non-main eigenvalue

An eigenvalue μ of a graph G is called *non-main* if $\mathcal{E}_G(\mu)$ is orthogonal to the all one vector $\mathbf{j}$. Otherwise it is called *main*.

Using the notation of Theorem 25, we have the following.

Theorem 27 *Let G be a graph with eigenvalue μ. Then μ is a non-main eigenvalue of G if and only if for all $u \in V(G)$, $<\mathbf{j}, \mathbf{s}_u>= -1$.*

Proof Using the basis given in Theorem 25, the assertion easily follows. □

Theorem 28 *Let G be an r-regular graph of order n and with eigenvalue μ. Then $<\mathbf{j}, \mathbf{j}>= n/(\mu - r)$.*

Proof Use $< \sum \mathbf{s}_u, \mathbf{j} >= -n$. $\square$

Theorem 29 *Let G be a graph with star set X and star complement H for eigenvalue μ. Then μ is a non-main eigenvalue of G if and only if μ is a non-main eigenvalue of $H+u$ for all $u \in X$.*

Proof We use the basis given in Theorem 25. Every eigenvector of this basis corresponds to an eigenvector of $H+u$ for some unique $u \in X$. Therefore the assertion easily follows. $\square$

23 Bound on the order of graphs

The following inequality is proved using a well known technique in algebraic combinatorics: Consider a vector space of polynomials of dimension n and find a set of m independent polynomials. Then $m \leq n$.

Theorem 30 *Let $G \neq K_2, 2K_2$ be a graph of order n and of μ-rank $t > 1$ for eigenvalue $\mu \neq -1, 0$. Then*

$$n \leq \binom{t+1}{2},$$

and hence a star set has size at most $\binom{t}{2}$.

Proof Let

$$A = \begin{pmatrix} A_X & B^T \\ B & C \end{pmatrix},$$

where A and A_X are the adjacency matrices of A and the subgraph induced by a star set X, respectively. Let $S = [B \quad C - \mu I]$ and let $\mathbf{s}_u$ denote the columns of S. Define polynomials $F_u : \mathbb{R}^t \to \mathbb{R}$ as

$$F_u(\mathbf{x}) =< \mathbf{s}_u, \mathbf{x} >^2$$

which are quadratic homogenous functions of the general form $f(x_1, x_2, \ldots, x_t) = \sum c_i x_i^2 + \sum d_{ij} x_i x_j$. If we prove that F_u are independent, then we have $n \leq t + \binom{t}{2} = \binom{t+1}{2}$.

First we suppose that G is connected and μ is not its index. Let $\mathbf{b} = (b_1, b_2, \ldots, b_n)^T$ be the positive eigenvector corresponding to the index. Let $\mathbf{b}' = (b_{n-t+1}, b_2, \ldots, b_n)^T$. Assume that $\sum \alpha_u F_u = 0$. Let $\mathbf{a} = (\alpha_1, \alpha_2, \ldots, \alpha_n)^T$ and $\mathbf{a}' = (\alpha_1 b_1, \alpha_2 b_2, \ldots, \alpha_n b_n)^T$. We have

$$A\mathbf{a} = -\mu^2 \mathbf{a}$$

and

$$A\mathbf{a}' = \mu\mathbf{a}'.$$

Since $\mu \neq -\mu^2$, we have $\mathbf{a}^T\mathbf{a}' = 0$, i. e $\sum \alpha_u^2 b_u = 0$ and so $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. The first relation follows from $\sum \alpha_u F_u(\mathbf{x}) = 0$ by letting $\mathbf{x} = \mathbf{s}_i$ and the second relation follows from $\sum \alpha_u F_u(\mathbf{x}) = 0$ by letting $\mathbf{x} = \mathbf{s}_i + \mathbf{b}'$ (note that $< \mathbf{s}_u, \mathbf{b}' >= -b_u$ by the reconstruction Theorem).

The rest of proof is straightforward. □

Remark The above bound is sharp since it is attained in the following examples: (i) The unique exceptional graph of order 36 (here $\mu = -2$ and $t = 8$) which is obtained from $L(K_8)$ by switching with respect to the vertices in a clique of size 8, (ii) K_3, $\mu = 2$ and (ii) P_3, $\mu = \pm\sqrt{2}$.

Remark A strongly regular graph attaining the absolute bound is called *extremal*. The only known extremal strongly regular graphs are pentagon, complete multipartite graphs, the Schlafli graph (27 vertices), the McLaughlin graph (275 vertices) and the complement of these two latter graphs. It is a long time standing question whether these are the only examples.

Remark The bounds given in this section extend the absolute bound (which is equal to the bound given in Theorem 32) for regular graphs and any graph in general.

Theorem 31 *Let G be a graph of order n and of μ-rank $t > 2$ for a non-main eigenvalue $\mu \neq -1, 0$. Then*

$$n \leq \binom{t+1}{2} - 1 = \frac{1}{2}(t-1)(t+2).$$

Proof We use the notation in the proof of Theorem 30. Let

$$F(\mathbf{x}) =< \mathbf{j}, \mathbf{x} >^2 .$$

We show that F is not in the span of F_u. Assume by the contrary that $F = \sum \beta_u F_u$. We have

$$< \mathbf{j}, \mathbf{x} >< \mathbf{j}, \mathbf{y} >= \sum \beta_u < \mathbf{s}_u, \mathbf{x} >< \mathbf{s}_u, \mathbf{y} > .$$

First let $\mathbf{x} = \mathbf{y} = \mathbf{s}_i$ to find

$$\mathbf{j} = (\mu^2 I + A)\mathbf{b},$$

where $\mathbf{b} = (\beta_1, \ldots, \beta_n)$. Then let $\mathbf{x} = \mathbf{s}_i$ and $\mathbf{y} = \mathbf{j}$ to find

$$< \mathbf{j}, \mathbf{j} > \mathbf{j} = (\mu I - A)\mathbf{b}.$$

From these we see that $\mathbf{b}$ is a constant value vector. Let $\beta_u = \beta$. Now

$$\beta^2 (\sum < \mathbf{s}_u, \mathbf{x} >< \mathbf{s}_u, \mathbf{y} >)^2 =< \mathbf{j}, \mathbf{x} >^2 < \mathbf{j}, \mathbf{y} >^2 = \beta^2 \sum < \mathbf{s}_u, \mathbf{x} >^2 \sum < \mathbf{s}_u, \mathbf{y} >^2 .$$

From the Cauchy-Schwarz inequality, we have $< \mathbf{s}_u, \mathbf{x} >= \alpha < \mathbf{s}_u, \mathbf{y} >$ and so $\mathbf{x} = \alpha \mathbf{y}$ which means that $t = 1$. □

Theorem 32 *Let G be a r-graph of order n and of μ-rank $t > 2$ for an eigenvalue $\mu \neq -1, 0, r$. Then*

$$n \leq \binom{t+1}{2} - 1 = \frac{1}{2}(t-1)(t+2).$$

Moreover, equality holds if and only if G is an extremal strongly regular graph with μ as its eigenvalue of greatest multiplicity.

Proof The first part follows from Theorem 31. For the second part, a proof is given in [14]. We here give an easier proof. Define $f(\mathbf{x}) =< \mathbf{x}, \mathbf{x} >$. By the proof of Theorem 31, f is in the span of $F_u(\mathbf{x}) =< \mathbf{s}_u, \mathbf{x} >$ and $F(\mathbf{x}) =< \mathbf{j}, \mathbf{x} >$. So we may write $f = \delta F + \sum_u \varepsilon_u F_u$. We have $< \mathbf{x}, \mathbf{y} >= \delta < \mathbf{j}, \mathbf{x} >< \mathbf{j}, \mathbf{y} > + \sum_u \varepsilon_u < \mathbf{x}, \mathbf{s}_u >< \mathbf{y}, \mathbf{s}_u >$. If we put $\mathbf{x} = \mathbf{s}_i$ and $\mathbf{y} = -\mathbf{j}$, then we obtain that $(\mu I - A)\varepsilon$ is a constant vector, where $\varepsilon^t = (\varepsilon_1, \ldots, \varepsilon_n)$. If we evaluate f in $\mathbf{s}_i$, then we obtain that $(\mu^2 I + A)\varepsilon$ is a constant vector. This shows that ε_u are constant. Now if put $\mathbf{x} = \mathbf{s}_i$ and $\mathbf{y} = \mathbf{s}_j$, then we find that any two vertices have a constant number of common neighbors. □

Remark For $\mu = 0$, consider $(n-t)K_1 \cup K_t$ and for $\mu = -1$, consider $(t-1)K_1 \cup K_{n-t+1}$. This shows that in these cases n is not bounded by t.

24 An application: Graph construction

Sometimes the star complement technique can be used to construct graphs with given spectral (or topological) properties. In this approach, first the star complement is guessed and then it is possibly extended to a whole graph using the compatibility graph.

We present two successful applications of this method. One of the main achievements in this direction was the classification of all maximal exceptional graphs (473 graphs in total). A thorough treatment of the problem can be found in [11] and [15]. We only make some comments. An exceptional graph is a connected graph with least eigenvalue at least -2 which is not a generalized line graph. It was proved many years ago that such a graph has at most 36 vertices and has degree at most 28. All regular exceptional graphs, 187 in number, were found very soon. Also exceptional graphs with least eigenvalue greater than -2 (573 graphs) were known in early years. However, the general problem remained open until it was answered recently in [17], where all maximal exceptional graphs were found. Note that any exceptional graph is an induced subgraph of a maximal exceptional graph. The classification was done by the star complement technique making an extensive use of computer. The key observation was that any exceptional graph has an exceptional star complement with least eigenvalue greater than -2.

The other application is on extremal strongly regular graphs. Rowlinson [6] investigates such graphs having a maximum number of independent vertices (obtained from Cvetković's Theorem). He proves that such a graph is necessarily the Schlafli graph or the McLaughlin graph.

25 The general and restricted problems

The general problem is to find all graphs having a given graph as a star complement. In other words, in the reconstruction theorem (Theorem 25), given C, we want to find A_X, B, μ. The restricted problem is to find all graphs having a given graph as a star complement for the eigenvalue μ. In other words, in the reconstruction theorem, given C and μ, we want to find all A_X and B.

Why are these problems interesting? We can give three reasons: First it is a natural curiosity to try to identify the maximal graphs corresponding to nice star complements (such as complete, path, cycle, tree,...). Secondly, the maximal graphs obtained in this way might be objects with interesting properties. Thirdly, the maximal graphs obtained in this way provide lower bounds on the maximum order of a graph of a given μ-rank.

The known results are as follows. In the following, H is used to denote a star complement.

25.1 Graphs with small μ-rank

The general problem has been answered for all graphs with at most 5 vertices and $\mu \neq -1, 0$ [25]. All maximal reduced graphs with rank at most 7 are obtained in [41]. In [1], the number of all reduced graphs with rank at most 7 is given. For rank 8, all *maximum* reduced graphs are found [1].

25.2 Independence set

For $H = \overline{K_t}$, there is a simple characterization given in [32].

25.3 Star

For $H = K_{1,s}$, there is a thorough discussion in [32]. Also for regular extensions, we refer to [23]. The case $K_{1,5}$ is teated in full generality in [24]. See also [26].

25.4 Complete bipartite graphs

For $H = K_{r,s}$, some general observations are given in [30]. Also in this reference, the case $K_{2,5}$ is treated in detail. All these and the case $H = K_{r,s} + tK_1$ with a general discussion on μ, are also available at [26].

25.5 Star+isolated vertices

For $H = K_{1,s} + 2K_1$ ($\mu = 1$) and $H = \overline{K_{1,s} + 2K_1}$ ($\mu = -2$), the maximal graphs are found in [29]. In this reference, also the similar problem has been solved for H being a graph consisting of an isolated vertex together $K_{2,s}$ with a pendant edge at a vertex of degree s. The case $H = K_{1,16} + 6K_1$ for $\mu = 2$ is done in [26] (also presented in [16]).

25.6 Union of two paths or a path and a cycle

For $H = P_r$, $H = P_r + P_s$ and $H = P_r + C_s$ and $\mu = -2$, the results are given in [13]. Note that these types of star complements are more difficult and involved than those of types in the previous subsections.

25.7 Union of odd cycles

For $H = \bigcup_i C_{2s_i+1}$, there is a complete answer in [27] and [8].

25.8 Complete graphs

For $H = K_8$ and $\mu = -2$, it is shown (using a computer search) that there are exactly 363 maximal graphs [10].

25.9 Line graph of trees

Let T be tree. For $H = L(T)$ and $\mu = -2$, the results are given in [11]. In this reference, there are also results for line graph of odd-unicyclic graphs.

25.10 Trees, unicyclic and complete graphs with 1 as the second largest eigenvalue

See (Stanić, 2007 [4]) and (Stanić and Simić, [2]).

26 Other problems

26.1 Matching type problem

Sometimes it is interesting to find A_X, μ, C for given B in Theorem 25. An example is a "matching type problem", where B is assumed to be the identity matrix [21].

26.2 Extremal strongly regular graphs

The only known extremal strongly regular graphs are pentagon, complete multipartite graphs, the Schlafli graph, the McLaughlin graph and the complement of these two latter graphs. It is a long time standing question whether these are the only examples.

26.3 (-1)-rank

Can we make better the bound given in Theorem 17 for $m(-1,t)$? What can be said about the maximal elements of $\mathcal{G}(-1,t)$?

26.4 Characterization of the well known graphs by star complement

For example Kneser graph $\mathrm{KG}(n,2)$ is the unique regular graph with the star complement $K_{1,n-3} + 2K_1$ for the eigenvalue 1. There are many results of this type. See for example [29]. For the Hoffman-Singleton graph, see [3].

26.5 rank

Prove the conjecture for $m(0,t)$. Give a characterization of maximal elements in $\mathcal{G}(0,t)$. Give a characterization of minimal elements in $\mathcal{G}(0,t)$. Give a characterization of minimal-maximal elements in $\mathcal{G}(0,t)$.

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The following is a complete (?) list of papers which deal with the star complement technique. The entries are sorted by date.

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