A note on the spectral characterization of θ-graphs

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Abstract

We consider θ-graphs, that is, graphs obtained by subdividing the edges of the multigraph consisting of 3 parallel edges. It is shown that any θ-graph G is determined by the spectrum (the multiset of eigenvalues) except possibly when it contains a unique 4-cycle.

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1 Introduction

In this paper, we are concerned only with undirected simple graphs (loops and multiple edges are not allowed). Let G be a graph with the adjacency matrix A. We denote $\det(\lambda I - A)$, the characteristic polynomial of G, by $P(G, \lambda)$. The multiset of eigenvalues of A is called the *adjacency spectrum*, or simply the *spectrum* of G. Since A is a symmetric matrix, the eigenvalues of G are real. Two nonisomorphic graphs with the same spectrum are called *cospectral*. We say that a graph is *determined by the spectrum* (*DS* for short) if there is no other nonisomorphic graph with the same spectrum.

In [4], it is conjectured that almost all graphs are DS. Nevertheless, the set of graphs which are known to be DS is small and therefore it would be interesting to find more examples of DS graphs. For a survey of the subject, the reader can consult [4, 5]. A list of more recent papers which have not been cited in [4, 5] includes [1, 2, 10, 11]. In recent years, spectral characterization of some well known classes of graphs possessing simple structures such as starlike trees [9, 14], lollipop graphs [2, 8], the complement of the path [6], graphs with index at most 2 [11, 13] and

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$\sqrt{2+\sqrt{5}}$ [7] have been studied. Here, we continue this line of research by investigating the so called θ-graphs. Let P_n and C_n denote the path and the cycle with n vertices, respectively. We denote the graph shown in Figure 1 (2) by $\theta(a,b,c)$ ($d(a,b,c)$) and call it a θ-graph (d-graph). Note that in both graphs removing the vertices of degree 3, leaves three disjoint paths P_a, P_b and P_c. For a $\theta(a,b,c)$ graph, we always assume that $a \leq b \leq c$ and for a $d(a,b,c)$ graph, $a \leq c$. In this note, we show that any θ-graph G is determined by the spectrum (the multiset of eigenvalues) except possibly when it contains a unique 4-cycle.

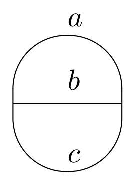

Figure 1: $\theta(a,b,c)$

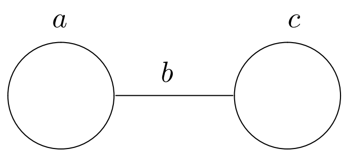

Figure 2: $d(a,b,c)$

2 Structure of graphs cospectral to θ-graphs

In this section we determine the structure of graphs which can be cospectral to a θ-graph with no 4-cycle. The following lemma shows that the degree sequence of such graphs is determined by the spectrum. In order to prove this, we use the fact that two cospectral graphs have the same number of closed walks for any length [4]. Let G and H be two cospectral graphs. Then the degrees of vertices satisfy certain equations. Let x_i and y_i denote the numbers of vertices of degree i in G and H, respectively. By counting the number of vertices, edges and closed walks of length 4 in G and H, we have the following relations:

$$\sum x_i = \sum y_i,$$
$$\sum ix_i = \sum iy_i,$$
$$\sum ix_i + 4\sum \binom{i}{2} x_i + 8n_4 = \sum iy_i + 4\sum \binom{i}{2} y_i + 8n_4',$$

where n_4 and n'_4 are the numbers of 4-cycles in G and H, respectively. By adding up these equations with coefficients $1, -5/4$ and $1/4$, respectively, we obtain that

$$\sum \binom{i-1}{2} x_i + 2n_4 = \sum \binom{i-1}{2} y_i + 2n'_4. \tag{1}$$

Lemma 1 *The degree sequence of any graph H cospectral to a θ-graph G with n vertices and with no 4-cycle is determined by the shared spectrum.*

Proof. Let y_i denote the number of vertices of degree i in H. Then by (1),

$$\sum \binom{i-1}{2} y_i + 2n'_4 = 2,$$

where n'_4 is the number of cycles of length 4 in H. This yields

$$y_0 + y_3 + 2n'_4 = 2$$

and $y_i = 0$ for $i > 3$. If $n'_4 = 1$, then $y_0 = y_3 = 0$ and so $y_2 = n + 2$, a contradiction. Therefore, $n'_4 = 0$ and we have $y_0 + y_3 = 2$. If $y_0 = 2$ and $y_3 = 0$, then $y_2 = n + 4$, a contradiction. If $y_0 = y_3 = 1$, then $y_2 = n + 1$ which is impossible. Hence $y_0 = 0$ and $y_3 = 2$ which imply the assertion. □

The following is a direct consequence of the lemma above and the fact that cycles have an eigenvalue 2.

Lemma 2 *A graph cospectral to a θ-graph with no 4-cycle and no eigenvalue 2 is θ-graph or d-graph with no 4-cycle.*

3 No θ-graphs are cospectral

In the section we show that no two θ-graphs are cospectral. To do this, we first need to compute the characteristic polynomial of θ-graphs. We make use of the following lemma.

Lemma 3 [3, 12] *Let v be a vertex of a graph G and let $\mathcal{C}(v)$ denote the collection of cycles containing v. Then the characteristic polynomial of G satisfies*

$$P(G, \lambda) = \lambda P(G \setminus \{v\}, \lambda) - \sum_{u \sim v} P(G \setminus \{u, v\}, \lambda) - 2 \sum_{Z \in \mathcal{C}(v)} P(G \setminus V(Z), \lambda).$$

For the sake of simplicity, we denote $P(P_r, \lambda)$ by $p_r = p_r(\lambda)$. By convection, we let $p_0 = 1$, $p_{-1} = 0$ and $p_{-2} = -1$. Using Lemma 3 with v being the vertices of degree 3, we can compute the characteristic polynomial of $\theta(a,b,c)$ in terms of the characteristic polynomial of paths. We have

$$\begin{aligned}P(\theta(a,b,c),\lambda) =&\lambda^2 p_a p_b p_c - 2\lambda(p_{a-1}p_b p_c + p_a p_{b-1} p_c + p_a p_b p_{c-1})\\&+ 2(p_{a-1}p_{b-1}p_c + p_{a-1}p_b p_{c-1} + p_a p_{b-1} p_{c-1})\\&+ p_{a-2}p_b p_c + p_a p_{b-2} p_c + p_a p_b p_{c-2} - 2(p_a + p_b + p_c).\end{aligned} \tag{2}$$

The next lemma follows from (2) and the fact that $p_r(2) = r + 1$.

Lemma 4 $P(\theta(a,b,c),2) = (a-1)(b-1)(c-1) - 4(a+b+c+1)$.

By Lemma 3, we have

$$p_r = \lambda p_{r-1} - p_{r-2}.$$

Solving this recurrence equation, we find that for $r \geq -2$,

$$p_r = \frac{x^{2r+2} - 1}{x^{r+2} - x^r}, \tag{3}$$

where x satisfies $x^2 - \lambda x + 1 = 0$. If we substitute (3) in (2), then we obtain

$$(x^2-1)^3 x^{m+2} P(\theta(a,b,c),\lambda) + 1 - 4x^2 + 4x^4 - x^{2m+6}(x^2-2)^2 = Q(a,b,c;x), \tag{4}$$

where $m = a + b + c$ and

$$\begin{aligned}Q(a,b,c;x) =&x^{2a+6} + x^{2b+6} + x^{2c+6} + 2x^{a+b+2} + 2x^{a+c+2} + 2x^{b+c+2} - 4x^{a+b+4} - 4x^{a+c+4} - 4x^{b+c+4}\\&+ 2x^{a+b+6} + 2x^{a+c+6} + 2x^{b+c+6} - x^{2a+2b+4} - x^{2a+2c+4} - x^{2b+2c+4} + 4x^{2a+b+c+6}\\&+ 4x^{a+2b+c+6} + 4x^{a+b+2c+6} - 2x^{2a+b+c+4} - 2x^{a+2b+c+4} - 2x^{a+b+2c+4}\\&- 2x^{2a+b+c+8} - 2x^{a+2b+c+8} - 2x^{a+b+2c+8}.\end{aligned} \tag{5}$$

(We have used Maple to perform the calculations).

Lemma 5 *No two nonisomorphic θ-graphs are cospectral.*

Proof. Suppose that $G = \theta(a,b,c)$ and $G' = \theta(a',b',c')$ are cospectral. By the convection, $a \leq b \leq c$ and $a' \leq b' \leq c'$. since G and G' have the same number of vertices, we have

$$a + b + c = a' + b' + c', \tag{6}$$

and by (4),

$$Q(a,b,c;x) = Q(a',b',c';x). \tag{7}$$

Also by Lemma 4,

$$(a-1)(b-1)(c-1) = (a'-1)(b'-1)(c'-1). \tag{8}$$

The smallest exponent of x in $Q(a,b,c;x)$ is equal to $2a+6$ or $a+b+2$. Therefore, by (7), without loss of generality, we may assume that one of the following occurs: (i) $2a+6 = 2a'+6$, (ii) $a+b+2 = a'+b'+2$ or (iii) $2a+6 = a'+b'+2$.

First let (i) hold. Then $a = a'$. If $a \neq 1$, then by (6) and (8), $(a, b, c) = (a', b', c')$ and we are done. Hence suppose that $a = 1$. The smallest power of x is equal to $b + 3$ and $b' + 3$ in $Q(a, b, c; x) - x^{2a+6}$ and $Q(a', b', c'; x) - x^{2a+6}$, respectively. Thus, $b = b'$ and the assertion follows from (6).

Next suppose that (ii) holds. Then by (6), $c = c'$. If $c \neq 1$, then by (6) and (8), $(a, b, c) = (a', b', c')$. If $c = 1$, then we necessarily have $b = b' = 1$. Therefore, $(a, b, c) = (a', b', c') \in \{(0, 1, 1), (1, 1, 1)\}$ and the assertion follows.

Finally we assume that (iii) holds. We may suppose that (ii) does not occur. Then since (ii) doss not hold, we have $a + b + 2 > 2a + 6$ which yields $2b + 6 > 2a + 6$. Hence the coefficient of x^{2a+6} in the left hand side of (7) is 1. This provides a contradiction since the coefficient of $x^{a'+b'+2} = x^{2a+6}$ in the right hand side of (7) is at least 2. $\square$

4 θ- and d-graphs are not cospectral

In the section we demonstrate that a θ-graph and a d-graph cannot be cospectral. Using Lemma 3, we first compute the characteristic polynomial of d-graphs. We have

$$\begin{aligned} P(d(r, k, s), \lambda) =& \lambda^2 p_r p_s p_k - 2\lambda(p_{r-1} p_s p_k + p_r p_{s-1} p_k + p_r p_s p_{k-1} + p_r p_k + p_s p_k) \\ &+ 2(2p_{r-1} p_{s-1} p_k + p_{r-1} p_s p_{k-1} + p_r p_{s-1} p_{k-1}) + p_r p_s p_{k-2} \\ &+ 4(p_{r-1} p_k + p_{s-1} p_k) + 2(p_r p_{k-1} + p_s p_{k-1}) + 4p_k. \end{aligned} \tag{9}$$

If we substitute (3) in (9), then we obtain

$$(x^2 - 1)^3 x^{m+2} P(d(r, k, s), \lambda) + 1 - 4x^2 + 4x^4 - x^{2m+6}(x^2 - 2)^2 = U(r, k, s; x), \tag{10}$$

where $m = r + s + k$ and

$$\begin{aligned} U(r, k, s; x) =& 2x^{r+1} + 2x^{s+1} - 6x^{r+3} - 6x^{s+3} + 4x^{r+5} + 4x^{s+5} - x^{2r+2} - x^{2s+2} + 2x^{2r+4} + 2x^{2s+4} \\ &- 4x^{r+s+2} + 8x^{r+s+4} - 4x^{r+s+6} + 2x^{r+2s+3} + 2x^{2r+s+3} - 2x^{r+2s+5} - 2x^{2r+s+5} - x^{2r+2s+4} \\ &+ 2x^{r+2k+5} + 2x^{s+2k+5} - 2x^{r+2k+7} - 2x^{s+2k+7} - 2x^{2r+2k+6} - 2x^{2s+2k+6} + x^{2r+2k+8} \\ &+ x^{2s+2k+8} + 4x^{s+r+2k+4} - 8x^{s+r+2k+6} + 4x^{s+r+2k+8} - 4x^{r+2s+2k+5} - 4x^{s+2r+2k+5} \\ &+ 6x^{r+2s+2k+7} + 6x^{s+2r+2k+7} - 2x^{r+2s+2k+9} - 2x^{s+2r+2k+9} + x^{2k+6}. \end{aligned} \tag{11}$$

The following lemma follows from (9) and $p_r(2) = r + 1$.

Lemma 6 $P(d(r, k, s), 2) = (r + 1)(s + 1)(k - 1)$.

Lemma 7 *Let $a \leq b \leq c$, $r = a + b + 1 \leq s = c - a - 1$ and $k = a$. Then $P(\theta(a, b, c), 2) \neq P(d(r, k, s), 2)$.*

Proof. Let $h := P(\theta(a,b,c),2) - P(d(r,k,s),2)$. Then by Lemmas 4 and 6,

$$h = (a+1)(a^2 - ac + ab - c - 5 - 3b).$$

By the assumption, we have $c \geq 2a + b + 2$. Therefore,

$$\begin{aligned} h &= (a+1)(a^2 - ac + ab - c - 5 - 3b) \\ &\leq (a+1)(a^2 - a(2a+b+2) + ab - (2a+b+2) - 5 - 3b) \\ &= -(a+1)(a^2 + 4a + 4b + 7) \\ &< 0. \end{aligned}$$

□

Lemma 8 *There is no θ-graph cospectral with a d-graph.*

Proof. Let $G = \theta(a,b,c)$ be cospectral with $G' = d(r,k,s)$. By the convection, $a \leq b \leq c$ and $2 \leq r \leq s$. Since G and G' have the same number of vertices, we have

$$a + b + c = r + s + k, \tag{12}$$

and by (4) and (10),

$$Q(a,b,c;x) = U(r,k,s;x). \tag{13}$$

We claim that $r = a + b + 1$ and $k = a$. Note that if this claim is proven, then by Lemma 7, we have a contradiction and hence the assertion follows.

Let f denote the smallest exponent of x in $Q(a,b,c;x)$ (also in $U(r,k,s;x)$ by (13)). By (5), $f = 2a + 6$ or $a + b + 2$. Also by (11), $f = r + 1$ or $2k + 6$. We consider two cases.

(i) Let $f = 2a + 6 < a + b + 2$. It is easily seen that the coefficient of x^f in $Q(a,b,c;x)$ is 1. By (13), the coefficient of x^f in $U(a,b,c;x)$ should also be 1 and since the coefficient of x^{r+1} in $U(a,b,c;x)$ is at least 2, we necessarily have $f = 2k + 6$ which yields $k = a$. The smallest power of x is equal to $a + b + 2$ and $r + 1$ in $Q(a,b,c;x) - x^{2a+6}$ and $U(r,k,s;x) - x^{2k+6}$, respectively. Thus, $r + 1 = a + b + 2$ and the claim is established in this case.

(ii) Let $f = a+b+2 \leq 2a+6$. It is easily seen that the coefficient of x^f in $Q(a,b,c;x)$ is at least 2. By (13), the coefficient of x^f in $U(a,b,c;x)$ should also be at least 2 and since the coefficient of x^{2k+6} in $U(a,b,c;x)$ is 1, we necessarily have $f = r+1$ which yields $r+1 = a+b+2$. It remains to show that $k = a$. We first observe that if $b = 1$, then by Lemma 4, $P(\theta(a,b,c),2) < 0$ and so by Lemma 6, $k = 0$. This observation results in that if $b = 1$ and $a = 0$, the $k = a$ as required and so hereafter we may assume that $(a,b) \neq (0,1)$. First suppose $b \geq 2$ or $b = a = 1, c > 4$. By $a+b+1 = r \leq s \leq s+k \leq a+b+c-r = c-1$, we have $c \geq a+b+2$. We determine the smallest power h of x in $\mathcal{Q} := Q(a,b,c;x) - 2x^{a+b+2} + 4x^{a+b+4}$. Then by (5) and noting that $c \geq a+b+2$, h is the smallest power of x in $\mathcal{Q}' = x^{2a+6} + x^{2b+6} + 2x^{a+c+2} + 2x^{b+c+2} + 2x^{a+b+6} - x^{2a+2b+4}$. It is then seen that $h = 2a+6$ and the coefficient of x^h in $\mathcal{Q}'$ (which is the same as that in $\mathcal{Q}$) is positive and different from 2. Now we compute the smallest power h' of x in $\mathcal{U} := U(r,k,s;x) - 2x^{r+1} + 4x^{r+3}$. By (11), h' is the smallest power of x in $\mathcal{U}' = x^{2k+6} + 2x^{s+1} - 2x^{r+3} - 6x^{s+3}$ (note that since $r = a+b+1 > 1$, all the powers in $\mathcal{U} - \mathcal{U}'$ are greater than $r+3$). Since $h' = h > 0$ and the coefficient of $x^{h'}$ in $\mathcal{U}'$ (which is the same as that in $\mathcal{U}$) is not 2, clearly we have $h' = 2k+6$. Hence $h = 2a+6 = h' = 2k+6$ which gives $k = a$. Now let $b = a = 1$ and $c \le 4$. By the above observation, we have $k = 0$. By (12) and Lemmas 4 and 6, $r+s+2 = c+4$ and $(r+1)(s+1) = 4(c+3)$. However, these equations have no solutions for $c \le 4$. This completes the proof. □

5 θ-graphs with an eigenvalue 2

The following lemma shows that there are a handful of θ-graphs admitting 2 as an eigenvalue.

Lemma 9 *$\theta(a,b,c)$ has 2 as an eigenvalue if and only if (a,b,c) is as follow.*

#	1	2	3	4	5	6	7	8	9	10
a	2	2	2	2	2	3	3	3	4	5
b	6	7	8	9	11	4	5	7	4	5
c	41	23	17	14	11	19	11	7	9	5

Moreover, for these graphs, 2 is the second largest eigenvalue and has multiplicity 1.

Proof. Let 2 be an eigenvalue of $\theta(a,b,c)$. Then, by Lemma 4,

$$abc - ab - ac - bc - 3(a+b+c) - 5 = 0. \qquad (14)$$

Note that $1 < a < 6$, since otherwise,

$$6bc \le ab + ac + bc + 3(a+b+c) + 5 \quad \text{or} \quad bc \ge ab + ac + bc + 3(a+b+c) + 5,$$

which are both impossible. Solving the equation (14) for $a = 2,3,4,5$ gives the first part of the lemma. Since the spectral radius of θ-graphs is greater than 2, the second part follows by removing the vertices of degree 3 and applying the interlacing theorem. □

It is not hard to show that all graphs of Lemma 9 are DS. We prove this assertion for some cases. The proof for other cases is similar.

Let G be any of the graphs of Lemma 9. Let H be cospectral to G. By Lemmas 1, 5, 8 and 9, $H = K + C_m$, where K is a θ- or a d- graph with no 4-cycle and $m \neq 4$. First let $G = \theta(2,11,11)$. By corresponding an eigenvector it is easy to see that G does not admit -2 as an eigenvalue. Since even cycles have an eigenvalue -2, it follows that m is odd. It is well known that the length of shortest odd cycle in a graph and the number of such cycles is determined by the spectrum. The shortest odd cycle of G is of length 15 and there are two such cycles. Since m is odd, it follows that H has more than 30 vertices, a contradiction. Therefore, G is DS.

Next let $G = \theta(5,5,5)$. Since G is bipartite, so is H. We have $K = \theta(2a+1, 2b+1, 2c+1)$ or $K = d(2r+1, k, 2s+1)$. If $K = d(2r+1, k, 2s+1)$, then H has at least 18 vertices, a contradiction to the fact that G has 17 vertices. Hence, $K = \theta(2a+1, 2b+1, 2c+1)$. Note that if $c \leq 2$, then the largest eigenvalue of H will be greater than the largest eigenvalue of G [3], a contradiction. Since $m \geqslant 6$, $a = b = 0$ and $c = 3$ which contradicts the fact that H has no 4-cycle. Hence, G is DS.

Finally assume that $G = \theta(2,9,14)$. The shortest odd cycle of G is of length 13 and there is a unique such cycle. Note that m is odd, since -2 is not an eigenvalue of G. If K has an odd cycle, then H has at least 28 vertices, a contradiction since H has 27 vertices. Therefore, K is bipartite. It follows that $-\lambda$ is an eigenvalue of G, where λ is the largest eigenvalue of K (also G). Since G is connected, we find that G is bipartite, a contradiction.

We rely on the following lemma.

Lemma 10 *Any θ-graph with an eigenvalue 2 is DS.*

6 The main result

We first consider θ-graphs which contain 4-cycles. There are only two θ-graphs with more than one 4-cycle. They are $\theta(1,1,1)$ and $\theta(0,2,2)$. We prove that both graphs are DS. For θ-graphs with a unique 4-cycle, a list of possible degree sequences of cospectral mates is presented.

Lemma 11 *Let G be a θ-graph containing more than one 4-cycle. Then G is DS.*

Proof. First assume that G has three 4-cycles. Then $G = \theta(1,1,1)$, a bipartite graph. There exists only one bipartite graph with 5 vertices and 6 edges, i.e. $K_{2,3}$ which is isomorphic to $\theta(1,1,1)$. It follows that G is DS. Now suppose that G has exactly two 4-cycles. Then G is necessarily $\theta(0,2,2)$ which is bipartite. There exist exactly three bipartite graphs with 6 vertices and 7 edges. One of them is $\theta(0,2,2)$. The other two graphs are obtained from $\theta(1,1,1)$ by adding a pendant edge at a vertex of degree 2 and 3. Let H be any of these graphs. Let y_i denote the number of vertices of degree i in H. Then $(y_1, y_2, y_3, y_4) = (1,2,3,0)$ or $(1,3,1,1)$. If H is cospectral to G, then by (1), we must have $y_3 + 3y_4 + 6 = 6$, a contradiction. Therefore, G is DS. □

Now assume that G is a θ-graph with n vertices containing a unique 4-cycle. Let H be cospectral to G and let y_i denote the number of vertices of degree i in H. Then by (1), we have $y_0 + y_3 + 3y_4 + 2n'_4 = 4$, where n'_4 is the number of 4-cycles in H. This equation leads to three solutions for the degree sequence of H: $(y_0, y_1, y_2, y_3, y_4; n'_4) = (0,0,n-2,2,0;1), (0,1,n-$

$3,1,1;0),(0,2,n-6,4,0;0)$. If $(y_0,y_1,y_2,y_3,y_4;n'_4)=(0,0,n-2,2,0;1)$, then by Lemma 9, H is a θ-graph or a d-graph which is impossible by Lemmas 5 and 8. For the other two cases, we find many candidates for H which make the problem more involved and complicated. Finally, we mention that a similar problem, i.e. lollipop graphs with 4-cycles have been dealt with in the long paper [2].

Finally we present our main result.

Theorem 1 *Any θ-graph with no unique 4-cycle is DS.*

Proof. If G has an eigenvalue 2, then the assertion follows from Lemma 10. Otherwise, it follows from Lemmas 2, 5, 8, and 11. □

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