Weakly saturated subgraphs of random graphs[1]

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Let G and F be graphs, and let $H \subset G$ be a spanning subgraph of G. The graph H is called *weakly F-saturated in G*, if there exists a sequence of graphs $H = H_0 \subset \cdots \subset H_m = G$, where each H_i is obtained from H_{i-1} by adding an edge that belongs to a copy of F in H_i. In other words, all the edges of $G \setminus H$ can be recovered one by one in a way such that each edge creates a new copy of F. The smallest number of edges in a weakly F-saturated subgraph of G is called *the weak F-saturation number of G* and is denoted by $\mathrm{wsat}(G, F)$. This notion was first introduced by Bollobás in 1968 [3].

The exact value of $\mathrm{wsat}(K_n, K_s)$ (here, as usual, K_n is a complete graph on n vertices) was achieved by Lovász [9]. He showed that if $n \geq s \geq 2$, then

$$\mathrm{wsat}(K_n, K_s) = \binom{n}{2} - \binom{n-s+2}{2}.$$

The next natural graph to consider on the role of F is a complete bipartite graph $K_{s,t}$. However, the value of $\mathrm{wsat}(K_n, K_{s,t})$ for an arbitrary choice of parameters is still unknown. The most general result was obtained by Kalai [4] in 1985 and Kronenberg, Martins and Morrison [8] in 2020. They found the exact value only for balanced bipartite graphs: for $t \geq 2$ and $n \geq 3t - 3$

$$\mathrm{wsat}(K_n, K_{t,t}) = (t-1)(n+1-t/2),$$
$$\mathrm{wsat}(K_n, K_{t,t+1}) = (t-1)(n+1-t/2) + 1.$$

Moreover, in [8] general bounds for arbitrary choice of parameters s, t were also obtained:

$$\mathrm{wsat}(K_n, K_{s,t}) \leq (s-1)(n-s) + \binom{t}{2}, \quad t > s \geq 2, \ n \geq 2(s+t) - 3; \tag{1}$$

$$\mathrm{wsat}(K_n, K_{s,t}) \geq (s-1)(n-t+1) + \binom{t}{2}, \quad t > s \geq 2, \ n \geq 3t - 3. \tag{2}$$

For $s = 1$, it is straightforward to show that $\mathrm{wsat}(K_n, K_{1,t}) = \binom{t}{2}$. The case $s = 2$ appears to be much more sophisticated, but in [10], it was managed to solve it.

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Theorem 1 *For all integers $t \geq 3$ and $n \geq t+2$, the following hold.*

- *If t is odd or $n \geq 2t-1$, then* $\mathrm{wsat}(K_n, K_{2,t}) = n - 2 + \binom{t}{2}$*;*
- *If t is even and $n \leq 2t-2$, then* $\mathrm{wsat}(K_n, K_{2,t}) = n - 1 + \binom{t}{2}$*.*

As usual we denote by $G(n,p)$ the binomial random graph on the vertex set $[n] := \{1, \ldots, n\}$, where every pair of distinct $i, j \in [n]$ is adjacent with probability p independently of the others. Hereinafter, we say that some property holds *with high probability*, or *whp* for short, if its probability tends to 1 as $n \to \infty$. In 2017, Korándi and Sudakov [6] proved that, if $s \geq 3$, then $\mathrm{wsat}(K_n, K_s)$ is *stable*, i.e., for constant $p \in (0,1)$, whp

$$\mathrm{wsat}(G(n,p), K_s) = \mathrm{wsat}(K_n, K_s),$$

and ask about the possible threshold for this *stability property*. We have managed to prove that the threshold exists, and give nontrivial bounds for its value [2].

Theorem 2 *There exists c such that, if $p < cn^{-\frac{2}{s+1}}(\ln n)^{\frac{2}{(s-2)(s+1)}}$, then whp* $\mathrm{wsat}(G(n,p), K_s) \neq \mathrm{wsat}(K_n, K_s)$*. If $p > n^{-\frac{1}{2s-3}}(\ln n)^2$, then whp* $\mathrm{wsat}(G(n,p), K_s) = \mathrm{wsat}(K_n, K_s)$*.*

It is natural to ask about the existence of a graph F such that the stability property does not hold for some constant p. We conjecture that there is no such F.

Conjecture 1 *Let $p \in (0,1)$ be constant. Then, for every F, whp*

$$\mathrm{wsat}(G(n,p), F) = \mathrm{wsat}(K_n, F). \tag{3}$$

In the favor of this conjecture we have found a sufficient condition for the stability property [5]. Below, we denote by $\delta(H)$ the minimum degree of graph H. Without loss of generality, we set $V(K_n) = [n]$.

Theorem 3 *Let F be a graph without isolated vertices, and let $p \in (0,1)$, $C \geq \delta(F) - 1$ be constants. For every $n \in \mathbb{N}$, let H_n^0 be a weakly F-saturated subgraph of K_n containing a set of vertices $S_n^0 \subset [n]$ with $|S_n^0| \leq C$, such that every vertex from $[n] \setminus S_n^0$ is adjacent to at least $\delta(F) - 1$ vertices of S_n^0. Then whp there exists a subgraph $F_n \subset G(n,p)$ which is weakly F-saturated and F_n has* $\min\{|E(G(n,p))|, |E(H_n^0)|\}$ *edges.*

This theorem implies the following.

Corollary 1 *Let $p \in (0,1)$ be constant. For an arbitrary graph F without isolated vertices, whp the equality (3) holds if, for every $n \in \mathbb{N}$, there exists a minumum (having $\mathrm{wsat}(K_n,F)$ edges) weakly F-saturated subgraph of K_n with the property described in Theorem 3.*

Indeed, the condition in Theorem 3 immediately implies that whp $\mathrm{wsat}(G(n,p),F) \leq \mathrm{wsat}(K_n,F)$. Since whp every pair of vertices of $G(n,p)$ has at least $|V(F)|$ pairwise adjacent common neighbors [11], whp $G(n,p)$ is weakly (K_n,F)-saturated implying that the value of the weak saturation in $G(n,p)$ cannot be less that in K_n.

It is easy to show that Corollary 1 implies that whp the stability property (3) holds for $F = K_s$ and $F = K_{s,t}$ for all values of s,t (despite the fact that we do not know the exact values of $\mathrm{wsat}(K_n,K_{s,t})$). Note that due to (1) and (2) whp $\mathrm{wsat}(G(n,p),K_{s,t}) = (s-1)n + C(s,t)$ for some constant $C = C(s,t)$.

Despite the fact that for $F = K_s$ we still do not know the exact threshold for the stability property (even in the simplest case $s = 3$), we have found it for stars $F = K_{1,t}$ [5].

Theorem 4 *Let $t \geq 3$. Denote $p_t = n^{-\frac{1}{t-1}}[\ln n]^{-\frac{t-2}{t-1}}$.*

- *There exists $c > 0$ such that, if $\frac{1}{n^2} \ll p < cp_t$, then whp $\mathrm{wsat}(G(n,p),K_{1,t}) \neq \mathrm{wsat}(K_n,K_{1,t})$.*
- *There exists $C > 0$ such that, if $p > Cp_t$, then whp $\mathrm{wsat}(G(n,p),K_{1,t}) = \mathrm{wsat}(K_n,K_{1,t})$.*

Note that Theorem 4 does not cover the case $t = 2$ as well as $p = O(1/n^2)$, which are much easier to handle. First, if $t \geq 3$ and $p < \frac{Q}{n^2}$ for some $Q > 0$, then whp there are no copies of $K_{1,t-1}$ in $G(n,p)$, and so whp there is stability only if the number of edges of the entire graph is exactly $\binom{t}{2}$, which does not happen whp when $p \ll \frac{1}{n^2}$ and has probability bounded away both from 0 and 1 when $\frac{q}{n^2} < p < \frac{Q}{n^2}$ for some $0 < q < Q$. The case $t = 2$ is also easy. If $p > (1+\varepsilon)\frac{\ln n}{2n}$ for some $\varepsilon > 0$, then whp $\mathrm{wsat}(G(n,p),K_{1,2}) = \mathrm{wsat}(K_n,K_{1,2})$. If $\frac{1}{n^2} \ll p < (1-\varepsilon)\frac{\ln n}{2n}$, then whp $\mathrm{wsat}(G(n,p),K_{1,2}) \neq \mathrm{wsat}(K_n,K_{1,2})$. If $\frac{q}{n^2} < p < \frac{Q}{n^2}$ for some $0 < q < Q$, then

$$\mathsf{P}\Big[\mathrm{wsat}(G(n,p),K_{1,2}) = \mathrm{wsat}(K_n,K_{1,2})\Big] =$$

$$\mathsf{P}(G(n,p) \text{ contains exactly one edge}) + o(1) = \binom{n}{2} p(1-p)^{\binom{n}{2}-1} + o(1)$$

is bounded away both from 0 and 1. Finally, if $p \ll \frac{1}{n^2}$, then whp $\mathrm{wsat}(G(n,p),K_{1,2}) = 0 \neq \mathrm{wsat}(K_n,K_{1,2})$.

Let us finally note that the asymptotical version of Conjecture 1 is true.

Theorem 5 *For every constant $p \in (0,1)$ and every graph F, whp*

$$\mathrm{wsat}(G(n,p),F) = \mathrm{wsat}(K_n,F)(1+o(1)).$$

Let us sketch the proof. Fix a graph F and constant $p \in (0,1)$. First of all let us recall that (see [1]) there exists a constant c_F such that $\mathrm{wsat}(K_n,F) = (c_F+o(1))n$ and that $c_F > 0$ if and only if F does not contain vertices with degree 1. If F contains a vertex with degree 1, then it is easy to see that there exists a constant w_F such that whp

$$\mathrm{wsat}(G(n,p),F) = \mathrm{wsat}(K_n,F) = w_F.$$

Assume that F has $s \geq 3$ vertices, and none of them has degree 1. It is very well known [7] that $G(n,p)$ admits a clique factor with cliques of size $\log_{1/p} n$. In other words, whp there are disjoint sets $V_1,\ldots,V_m \subset [n]$ such that each V_i has size $v_i \in \{\lfloor \log_{1/p} n \rfloor, \lceil \log_{1/p} n \rceil\}$, and V_i induces a clique in $G(n,p)$. Using standard arguments, it can be shown that whp V_i may be chosen in a way such that the bipartite graphs with parts (V_i,V_{i+1}) are pseudorandom in the following sense: for all $i \in [m]$, there exist disjoint $S_i^1, S_i^2 \subset V_i$ satisfying

- $S_i^2 \sqcup S_{i+1}^1$ induce cliques in $G(n,p)$;
- every vertex from $V_{i+1} \setminus S_{i+1}^1$ has at least $s-2$ neighbors $v_1,\ldots,v_{s-2}$ in V_i such that each v_i is adjacent to all vertices of S_{i+1}^1.

Each induced subgraph $G(n,p)[V_i]$ contains a subgraph H_i with $\mathrm{wsat}(K_{v_i},F)$ edges. Consider the graph H obtained by the union of H_i and $m-1$ complete bipartite graphs with parts (S_i^2,S_{i+1}^1), $i \in [m-1]$. Note that H has at most

$$m(c_F+o(1))\log_{1/p} n + (m-1)(s-2)^2 = (c_F+o(1))n$$

edges, and that it is weakly F-saturated in $G(n,p)$. It remains to recall that whp $G(n,p)$ is weakly F-saturated in K_n, and so whp $\mathrm{wsat}(G(n,p),F) \geq \mathrm{wsat}(K_n,F)$.

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