Some generalizations of Caristies fixed point theorem with applications to fixed point theory and minimization problem

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In this talk, we first present some generalizations of Caristi's fixed point theorem. Then we give some applications to fixed point theory of weakly contractive set-valued maps and Takahashi-type minimization theorem.

Caristi's fixed point theorem [1], which is an extension of Banach's contraction principle, states that
any map $T : M \to M$ has a fixed point provided that (M, d) is a complete metric space and there exists a lower semi-continuous map $\phi : M \to [0, \infty)$ such that

$$d(x, Tx) \leq \phi(x) - \phi(Tx), \qquad \text{for every} \quad x \in M.$$

This general fixed point theorem has found many applications in nonlinear analysis. In 1991, Takahashi [8] proved the following minimization theorem: **let (M, d) be a complete metric space and let $f : M \to (-\infty, \infty]$ be a proper lower semicontinuous and bounded below function. Suppose that for each $\widehat{x} \in M$ with**

$$\inf_{x \in M} f(x) < f(\widehat{x})$$

there exists $x \in X$ with $x \neq \widehat{x}$ such that

$$d(\widehat{x}, x) \leq f(\widehat{x}) - f(x).$$

Then there exists $\overline{x} \in M$ such that

$$f(\overline{x}) = \inf_{x \in M} f(x).$$

It is well known that Takahashi's existence theorem and Caristi's fixed point theorem are equivalent. Our first aim in this talk is to prove some generalizations of Caristi's fixed point theorem. Then, we apply our generalized Caristi's fixed point theorems to prove a fixed point result for weakly contractive set-valued maps and an extension of Takahashi minimization theorem.

Throughout the paper, let Ψ be the class of all the maps $\psi : M \times M \to \mathbb{R}$ which satisfies the following conditions:

(i) there exists $\widehat{x} \in M$ such that $\psi(\widehat{x}, .)$ is bounded below and lower semi-continuous and $\psi(., y)$ is upper semi-continuous for each $y \in M$;

(ii) $\psi(x, x) = 0$, for each $x \in M$;

(iii) $\psi(x, y) + \psi(y, z) \leq \psi(x, z)$, for each $x, y, z \in M$.

Remark. *Let $\phi : M \to \mathbb{R}$ be a lower bounded, lower semi-continuous function and let*

$$\psi(x, y) = \phi(y) - \phi(x).$$

Then trivially $\psi \in \Psi$. But there are other functions, not of this form, that are in Ψ. Take, for instance, the function $\psi(x, y) = e^{-d(x,y)} - 1$.

Lemma. Let (M, d) be a complete metric space and $\psi \in \Psi$. Let $\gamma : [0, \infty) \to [0, \infty)$ is subadditive, i.e. $\gamma(x + y) \leq \gamma(x) + \gamma(y)$, for each $x, y \in [0, \infty)$, nondecreasing continuous map such that $\gamma^{-1}\{0\} = \{0\}$. Define the order $\prec$ on M by

$$x \prec y \Leftrightarrow \gamma(d(x, y)) \leq \psi(x, y),$$

for any $x, y \in M$. Then $(M, \prec)$ is a partial order set which has minimal elements.

Proof. It is straightforward to see that $(M, \prec)$ is a partial order set. To show that $(M, \prec)$ has minimal elements, we show that any decreasing chain has a lower bound. Indeed, let $(x_\alpha)_{\alpha \in \Gamma}$ be a decreasing chain, then we have

$$0 \leq \gamma(d(x_\alpha, x_\beta)) \leq \psi(x_\alpha, x_\beta)$$

$$\leq \psi(\hat{x}, x_\beta) - \psi(\hat{x}, x_\alpha).$$

Thus $(\psi(\widehat{x}, x_\alpha))_{\alpha\in\Gamma}$ is decreasing net of reals which is bounded below. Let (α_n) be an increasing sequence of elements from Γ such that

$$\lim_{n\to\infty} \psi(\widehat{x}, x_{\alpha_n}) = \inf\{\psi(\widehat{x}, x_\alpha); \alpha \in \Gamma\}.$$

Then for each $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that for each $m \geq n \geq n_\epsilon$ we have

$$\psi(\widehat{x}, x_{\alpha_n}) - \psi(\widehat{x}, x_{\alpha_m}) < \epsilon.$$

Hence for $m \geq n \geq n_\epsilon$

$$\gamma(d(x_{\alpha_m}, x_{\alpha_n}) \leq \psi(x_{\alpha_m}, x_{\alpha_n}) \leq$$

$$\psi(\widehat{x}, x_{\alpha_n}) - \psi(\widehat{x}, x_{\alpha_m}) < \epsilon.$$

Then our assumptions on γ imply that (x_{α_n}) is a Cauchy sequence and therefore converges to some $x \in M$ (note that (M, d) is complete). Since γ is continuous and $\psi(., x_{\alpha_n})$ is upper semi-continuous, then we have

$$\gamma(d(x, x_{\alpha_n})) = \limsup_{m\to\infty} \gamma(d(x_{\alpha_m}, x_{\alpha_n}) \leq$$

$$\limsup_{m\to\infty} \psi(x_{\alpha_m}, x_{\alpha_n}) \le \psi(x, x_{\alpha_n}).$$

This shows that $x \prec x_{\alpha_n}$ for all $n \ge 1$, which means that x is lower bound for $(x_{\alpha_n})_{n\ge1}$. In order to see that x is also a lower bound for $(x_\alpha)_{\alpha\in\Gamma}$, let $\beta \in \Gamma$ be such that $x_\beta \prec x_{\alpha_n}$ for all $n \ge 1$. Then for each $n \in \mathbb{N}$, we have

$$0 \le \gamma(d(x_\beta, x_{\alpha_n})) \le \psi(x_\beta, x_{\alpha_n})$$

$$\le \psi(\widehat{x}, x_{\alpha_n}) - \psi(\widehat{x}, x_\beta), \qquad (1)$$

Hence

$$\psi(\widehat{x}, x_\beta) \le \psi(\widehat{x}, x_{\alpha_n}), \quad \text{for all} \quad n \ge 1$$

which implies

$$\psi(\widehat{x}, x_\beta) = \inf\{\psi(\widehat{x}, x_\alpha); \alpha \in \Gamma\} = \lim_{n\to\infty} \psi(\widehat{x}, x_{\alpha_n}).$$

Thus from (1) we get $\lim_{n\to\infty} x_{\alpha_n} = x_\beta$, which implies $x_\beta = x$. Therefore, for any $\alpha \in \Gamma$, there exists $n \in \mathbb{N}$ such that $x_{\alpha_n} \prec x_\alpha$, i.e. x is a lower bound of $(x_\alpha)_{\alpha\in\Gamma}$ (note that $x \prec x_{\alpha_n}$). Zorn's lemma will therefore imply that $(M, \prec)$ has minimal elements.

In what follows, let $\mathcal{A}$ be the class of all maps $\eta : [0,\infty) \to [0,\infty)$ which are nondecreasing, continuous, and such that there exist $\epsilon_0 > 0$ such that

$$\eta(t) \leq \epsilon_0 \Rightarrow \eta(t) \geq \gamma(t),$$

where $\gamma : [0,\infty) \to [0,\infty)$ is a subadditive, nondecreasing continuous map such that $\gamma^{-1}\{0\} = \{0\}$.

Theorem. Let (M,d) be a complete metric space, $\eta \in \mathcal{A}$ and $\psi \in \Psi$. Define the order $\prec$ on M by

$$x \prec y \Leftrightarrow \eta(d(x,y)) \leq \psi(x,y).$$

Then $(M,\prec)$ has a minimal element x^*, i.e. if $x \prec x^*$ then we must have $x = x^*$.

Proof. Notice first that $(M,\prec)$ is not necessary a partial ordered set. Since $\psi \in \Psi$, there exists $\widehat{x} \in M$ such that $\{\psi(\widehat{x},x) : x \in M\}$ is bounded below. Set $\psi_0 = \inf\{\psi(\widehat{x}, x); x \in M\}$. For any $\epsilon > 0$, set

$$M_\epsilon = \{x \in M : \psi(\widehat{x}, x) \leq \psi_0 + \epsilon\}.$$

Since $\psi(\widehat{x}, .)$ is lower semi-continuous then M_ϵ is a closed nonempty subset of M. Also if $x, y \in M_\epsilon$ and $x \prec y$, then

$$\psi_0 \leq \psi(\widehat{x}, x) \leq \psi_0 + \epsilon$$

and

$$\psi_0 \leq \psi(\widehat{x}, y) \leq \psi_0 + \epsilon, \qquad (2)$$

also we have

$$\eta(d(x, y)) \leq \psi(x, y) \leq \psi(\widehat{x}, y) - \psi(\widehat{x}, x). \qquad (3)$$

From (2) and (3) we have $\eta(d(x, y)) \leq \epsilon$. Using ϵ_0 associated with η, we get

$$\gamma(d(x, y)) \leq \eta(d(x, y)) \leq \psi(x, y).$$

For on M_{ϵ_0} we define the new relation $\prec_*$ by

$$x \prec_* y \Leftrightarrow \gamma(d(x, y)) \leq \psi(x, y).$$

By Lemma, the partial order set $(M_{\epsilon_0}, \prec_*)$ has a minimal element x_*. Let us show that x_* is also a minimal element of $(M, \prec)$. Indeed let $x \in M$ be such that $x \prec x_*$. Then we have

$$0 \leq \eta(d(x, x_*)) \leq \psi(x, x_*) \leq \psi(\widehat{x}, x^*) - \psi(\widehat{x}, x). \quad (4)$$

Since $\psi(x_*, x) + \psi(x, x_*) \leq \psi(x_*, x_*) = 0$ and $\psi(x, x_*) \geq 0$ then $\psi(x_*, x) \leq 0$. Then from (4) we obtain

$$\psi(\widehat{x}, x) = (\psi(\widehat{x}, x) - \psi(\widehat{x}, x^*)) + \psi(\widehat{x}, x^*) \leq$$

$$\psi(\widehat{x}, x^*) \leq \psi_0 + \epsilon_0,$$

which implies that $\psi(\widehat{x}, x) \leq \psi_0 + \epsilon_0$; i.e. $x \in M_{\epsilon_0}$. As before, we have $\eta(d(x, x^*)) \leq \epsilon_0$ which implies that

$$\gamma(d(x, x^*)) \leq \eta(d(x, x^*)) \leq \psi(x, x^*)$$

which implies $x \prec_* x^*$. Since x_* is minimal in $(M_{\epsilon_0}, \prec_*)$ we get $x = x^*$. This completes the proof.

Corollary. *(***Generalized Caristi's fixed point theorem***)* **Let (M, d) be a complete metric space, $\eta \in \mathcal{A}$ and $\psi \in \Psi$. Let $T : M \to M$ be a map such that**

$$\eta(d(x, Tx)) \leq \psi(Tx, x), \quad \textbf{for all} \quad x \in M$$

(i.e. $Tx \prec x$). Then T has a fixed point.

Proof. By The previous theorem, the ordered set $(M, \prec)$ has a minimal element, say $\overline{x}$. By our assumption $T\overline{x} \prec \overline{x}$. Then by the minimality of $\overline{x}$, we get $T\overline{x} = \overline{x}$.

Remark. If we take $\psi(x, y) = \phi(y) - \phi(x)$, where $\phi : M \to [0, \infty)$ is lower semi-continuous and $\gamma(t) = ct$ then the above Corollary reduces to a result of Khamsi [4].

The proof of the above corollary yields the following endpoint result.

Corollary. **Let** (M, d) **be a complete metric space,** $\eta \in \mathcal{A}$ **and** $\psi \in \Psi$**. Let** $\mathcal{T} : M \multimap M$ **be a set-valued map such that** $\mathcal{T}x$ **is nonempty. If the condition**

$$\eta(d(x, y)) \leq \psi(y, x), \quad \text{for all} \quad y \in \mathcal{T}x$$

is satisfied, then T **has a endpoint in** M**, that is, there exists** $\overline{x} \in M$ **such that** $\mathcal{T}(\overline{x}) = \{\overline{x}\}$**.**

Applications.

Let (M, d) be a metric space. The map $T : M \to M$ is said to be *weakly contractive* if for all $x, y \in M$

$$d(Tx, Ty) \leq d(x, y) - \mu(d(x, y)),$$

where $\mu : [0, \infty) \to [0, \infty)$ is continuous, nondecreasing and $\mu^{-1}\{0\} = \{0\}$. In 2001 Rhodes [7] proved the following fixed point theorem for weakly contractive maps.

Theorem. Let (M, d) be a complete metric space and let $T : M \to M$ be a weakly contractive map. Then T has a unique fixed point.

In the following, let H denotes the Hausdorff metric on nonempty closed bounded subsets of

M. **The set-valued map $\mathcal{T} : M \multimap M$ with nonempty closed bounded values is said to be weakly contractive if for all $x, y \in M$**

$$\mathbf{H}(\mathcal{T}\mathbf{x}, \mathcal{T}\mathbf{y}) \leq \mathbf{d}(\mathbf{x}, \mathbf{y}) - \mu(\mathbf{d}(\mathbf{x}, \mathbf{y})),$$

where $\mu : [0, \infty) \to [0, \infty)$ is continuous, nondecreasing and $\mu^{-1}\{0\} = \{0\}$. In the light of the above Theorem, we pose the following problem:

Assume that $\mathcal{T} : M \multimap M$ is a weakly contractive set-valued map on a complete metric space (M, d) such that $\mathcal{T}x$ is nonempty closed and bounded for all $x \in M$. Does $\mathcal{T}$ has a fixed point?

In the following we give a partial solution of the above problem.

Theorem. Let (M, d) be a complete metric space and let $\mu \in \mathcal{A}$. Let $\mathcal{T} : M \multimap M$ be

a set-valued map with nonempty compact values satisfying

$$H(\mathcal{T}x, \mathcal{T}y) \le d(x,y) - \mu(d(x,y)), \quad (1)$$

for all $x, y \in M$. Then $\mathcal{T}$ has a fixed point.

Proof. Define $\nu(t) = \frac{2t-\mu(t)}{2}$ for $t \in [0, \infty)$, then $\nu(d(x,y)) \le d(x,y)$ for all $x, y \in M$. Therefore, given $x \in M$, the set

$$\{y \in \mathcal{T}(x) : \nu(d(x,y)) \le d(x, \mathcal{T}(x))\}$$

is nonempty (note that $\mathcal{T}(x)$ is compact). By the axiom of choice, there is a map $T : M \to M$ such that

$$\nu(d(x,Tx)) \le d(x, \mathcal{T}(x)) \quad \text{for} \quad x \in M.$$

By (1) we have,

$$d(Tx, \mathcal{T}(Tx)) \le H(\mathcal{T}x, \mathcal{T}(Tx)) \le$$

$$d(x,Tx) - \mu(d(x,Tx)). \quad (2)$$

Let $\phi(x) = d(x, \mathcal{T}x)$ for $x \in M$ and $\lambda(t) = t - \mu(t)$ for each $t \geq 0$. Then from (2) we get

$$\frac{\mu}{2}(d(x,Tx)) = \nu(d(x,Tx)) - \lambda(d(x,Tx) \leq$$

$$d(x,\mathcal{T}x) - d(Tx,\mathcal{T}(Tx)).$$

Thus

$$\frac{\mu}{2}(d(x,Tx)) \leq \phi(x) - \phi(Tx), \quad \text{for each} \quad x \in M,$$

and all the assumptions of our generalized Caristi's fixed point theorem are satisfied with $\psi(x,y) = \phi(y) - \phi(x)$(note that since $\mathcal{T}$ is continuous then ϕ is lower semicontinuous). Then T has a fixed point, say $\overline{x}$. Then $\overline{x} = T\overline{x} \in \mathcal{T}\overline{x}$.

The following result is an extension of Takahashi minimization theorem [8].

Theorem. (Takahashi-type Minimazation Theorem) Let (M,d) be a complete metric space, and $\eta \in \mathcal{A}$. Let $f : M \to (-\infty, \infty]$ be a

proper lower semicontinuous and bounded below function. Assume that for each $\widehat{x} \in M$ with $\inf_{z\in M} f(z) < f(\widehat{x})$, there exists $x \in M$ such that

$$x \neq \widehat{x} \qquad \textit{and} \qquad \eta(\mathbf{d}(\hat{\mathbf{x}}, \mathbf{x})) \leq \mathbf{f}(\hat{\mathbf{x}}) - \mathbf{f}(\mathbf{x}).$$

Then there exists $\overline{x} \in M$ such that

$$\mathbf{f}(\overline{x}) = \inf_{\mathbf{z}\in\mathbf{M}} \mathbf{f}(\mathbf{z}).$$

Proof. Define a set-valued map $\mathcal{T} : M \multimap M$ as

$$\mathcal{T}(x) = \{y \in M : \eta(d(x,y)) \leq f(x) - f(y)\}.$$

Let $\psi(y,x) = f(x) - f(y)$, then by the generalized Caristi's fixed point theorem, there exists $\overline{x} \in M$ such that $\mathcal{T}(\overline{x}) = \{\overline{x}\}$. By assumption, for all $\widehat{x} \in M$ there exists $x \in M$ such that $x \neq \widehat{x}$, we have $x \in \mathcal{T}(\widehat{x})$ and so $\mathcal{T}(\widehat{x}) \setminus \{\widehat{x}\} \neq \emptyset$ whenever $\inf_{z\in M} f(z) < f(\widehat{x})$. Hence we must have

$$f(\overline{x}) = \inf_{z\in M} f(z).$$

Vector Caristi's fixed point theorem
Let Y be a real Banach space. A nonempty subset P of Y is called *cone* if $cP \subseteq P$ for each $c \geq 0$. A cone P is called *pointed* if $P \cap (-P) = \{0\}$. It is easy to see that the relation

$$x \leq y \quad \text{if and only if} \quad y - x \in P$$

defines a partial ordering $\leq$ in Y, where P is a closed convex pointed cone. We shall write $x \ll y$ to indicate that $y - x \in \text{ int } P$.
The cone P is called *normal* if there is number $K > 0$ such that for all $x, y \in Y$,

$$0 \leq x \leq y \quad \Rightarrow \quad \|x\| \leq K\|y\|.$$

The least positive number satisfying above is called the *normal constant* of P. The cone P is called *regular* if every nondecreasing sequence which is bounded from above is convergent. That is, if $\{x_n\}$ is sequence such that

$$x_1 \leq x_2 \leq ... \leq x_n \leq ... \leq y,$$

for some $y \in Y$, then there is $x \in Y$ such that $\lim_{n\to\infty} \|x_n - x\| = 0$. Equivalently, the cone P is regular if and only if every nonincreasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone [3, Proposition 1.3.4].
By an ordered Banach space we mean a real Banach space $(Y, \|.\|)$ which is ordered by a pointed closed convex cone P with int $P \neq \emptyset$. In the following we always suppose Y is a Banach space, P is a pointed closed convex cone in Y with int $P \neq \emptyset$, and $\leq$ is partial ordering with respect to P.

Lemma. *([3, Proposition 1.3.2])* **If Y is an ordered Banach space with regular order cone, then each order bounded chain C of Y contains a nondecreasing (resp. a nonincreasing) sequence which converges strongly to $\sup C$ (resp. $\inf C$).**

Definition. *([2,5])* **Let X be a nonempty set and Y be an ordered Banach space with regular cone. Suppose that the mapping $d : X \times X \to Y$ satisfies***:*

(i) **0 $\leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;**

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(ii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Then (X, d) is called a cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. We say that $\{x_n\}$ is:

(i) **a Cauchy sequence if for every $c \in Y$ with $0 \ll c$, there exists $N \in \mathbb{N}$ such that, for all $\mathbf{m}, \mathbf{n} \geq \mathbf{N}$, $\mathbf{d}(\mathbf{x_m}, \mathbf{x_n}) \ll \mathbf{c}$***;*

(ii) **convergent to x and is denoted by $\lim_{n\to\infty} x_n = x$, if for every $c \in Y$ with $0 \ll c$, there exists $N \in \mathbb{N}$ such that, for all $n \geq N$, $d(x_n, x) \ll c$.**

The cone metric space (X, d) is called *complete*, if every Cauchy sequence in X is convergent.

Lemma. *([5])* **Let (X, d) be a cone metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $\lim_{n\to\infty} d(x_n, x) = 0$. Moreover, the limit of a convergent sequence is unique.**

Let (X, d) be a cone metric space and $A \subseteq X$. We say that A is a closed subset of X if for each convergent sequence $\{x_n\}$ in A with $\lim_{n\to\infty} x_n = x$ we have $x \in A$.

Lemma. *([5])* **Let** (X, d) **be a cone metric space. Let** $\{x_n\}$ **be a sequence in** X**. Then** $\{x_n\}$ **is a Cauchy sequence if and only if**

$$\lim_{\mathbf{m,n}\to\infty} \mathbf{d(x_n, x_m) = 0}.$$

Lemma. *([5])* **Let** (X, d) **be a cone metric space. Let** $\{x_n\}$ **and** $\{y_n\}$ **be two sequences in** X **and** $\lim_{n\to\infty} x_n = x$**,** $\lim_{n\to\infty} y_n = y$**. Then** $\lim_{\mathbf{n}\to\infty} \mathbf{d(x_n, y_n) = d(x, y)}$.

Definition. Let (X, d) **be a cone metric space and let** $\varphi : X \to Y$**. Then** φ **is said to be** P**-lower semicontinuous if for each** $\lambda \in Y$**, the set** $\{x \in X : \varphi(x) \leq \lambda\}$ **is closed in** X**.**

The following lemma will be used in the next section.

Lemma. Let (X, d) **be a cone metric space and let** $\varphi : X \to Y$ **be a** P**-lower semicontinuous map. Let** $\{x_n\}$ **be a convergent sequence with** $\lim_{n\to\infty} x_n = x$ **such that** $\{\varphi(x_n)\}$

is nonincreasing and convergent. Then

$$\varphi(\mathbf{x}) \leq \lim_{\mathbf{n}\to\infty} \varphi(\mathbf{x_n}).$$

Proof. Since $\varphi : X \to Y$ is P-lower semicontinuous then for each $n \in \mathbb{N}$, the set $B_n = \{z \in X : \varphi(z) \leq \varphi(x_n)\}$ is closed in X. Since $\{\varphi(x_n)\}$ is nonincreasing then $x_m \in B_n$ for each $m \geq n$. This along with the closeness of B_n imply that $x \in B_n$, that is, $\varphi(x) \leq \varphi(x_n)$ for each $n \in \mathbb{N}$. Therefore $\varphi(x) \leq \lim_{n\to\infty} \varphi(x_n)$.

Lemma. Let (X, d) be a complete cone metric space and $\varphi : X \to Y$ be a P-lower semicontinuous and lower bounded map. Define the order $\prec$ on X by

$$\mathbf{x} \prec \mathbf{y} \Leftrightarrow \mathbf{d}(\mathbf{x}, \mathbf{y}) \leq \varphi(\mathbf{y}) - \varphi(\mathbf{x}),$$

for any $x, y \in X$. Then $(M, \prec)$ is a partial order set which has minimal elements.

Proof. It is straightforward to see that $(M, \prec)$ is a partial order set. To show that $(M, \prec)$ has minimal elements, we show that any nonincrasing chain has a lower bound. Indeed, let $(x_\alpha)_{\alpha\in\Gamma}$ be a nonincreasing chain, then we have

$$0 \leq d(x_\alpha, x_\beta) \leq \varphi(x_\alpha) - \varphi(x_\beta).$$

Thus $(\varphi(x_\alpha))_{\alpha\in\Gamma}$ is a nonincreasing net in Y which is bounded below. By Lemma 1.1, there exist a nondecreasing sequence (α_n) of elements from Γ such that

$$\lim_{n\to\infty} \varphi(x_{\alpha_n}) = \inf\{\varphi(x_\alpha); \alpha \in \Gamma\}.$$

Then for each $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that for each $m \geq n \geq n_\epsilon$ we have $\|\varphi(x_{\alpha_m}) - \varphi(x_{\alpha_n})\| < \epsilon$. Since for $m \geq n$ we have

$$d(x_{\alpha_n}, x_{\alpha_m}) \leq \varphi(x_{\alpha_n}) - \varphi(x_{\alpha_m}) \qquad (1)$$

and every regular cone is normal, then there exists $K > 0$ such that

$$\|d(x_{\alpha_n}, x_{\alpha_m})\| \leq K\|\varphi(x_{\alpha_n}) - \varphi(x_{\alpha_m})\| < K.\epsilon$$

Thus $d(x_{\alpha_m}, x_{\alpha_n}) \to 0$ as $m, n \to \infty$. Therefore by Lemma , $\{x_{\alpha_n}\}$ is a Cauchy sequence in X and since (X, d) is a complete cone metric space then $\{x_{\alpha_n}\}$ converges to some $x^* \in X$. Then from (1) we have

$$\lim_{m\to\infty} d(x_{\alpha_n}, x_{\alpha_m}) \leq \lim_{m\to\infty} (\varphi(x_{\alpha_n}) - \varphi(x_{\alpha_m}))$$

Since φ is P-lower semicontinuous, then from Lemma we get

$$d(x_{\alpha_n}, x^*) \leq \varphi(x_{\alpha_n}) - \varphi(x^*)$$

This shows that $x^* \prec x_{\alpha_n}$ for all $n \geq 1$, which means that x^* is lower bound for $(x_{\alpha_n})_{n\geq 1}$. In order to see that x^* is also a lower bound for $(x_\alpha)_{\alpha\in\Gamma}$, let $\beta \in \Gamma$ be such that $x_\beta \prec x_{\alpha_n}$ for all $n \geq 1$. Then for each $n \in \mathbb{N}$, we have

$$0 \leq d(x_{\alpha_n}, x_\beta) \leq \varphi(x_{\alpha_n}) - \varphi(x_\beta) \qquad (2)$$

Hence

$$\varphi(x_\beta) \leq \varphi(x_{\alpha_n}), \quad \text{for all} \quad n \geq 1$$

which implies

$$\varphi(x_\beta) = \lim_{n\to\infty} \varphi(x_{\alpha_n}) = \inf\{\varphi(x_\alpha); \alpha \in \Gamma\}. \qquad (3)$$

Thus from (2) we get $\|d(x_{\alpha_n}, x_\beta)\| \leq K\|\varphi(x_{\alpha_n}) - \varphi(x_\beta)\|$, where K is normal constant of P. This along with (3) imply that

$$\lim_{n\to\infty} d(x_{\alpha_n}, x_\beta) = 0.$$

Then by Lemma 1.3, $\lim_{n\to\infty} x_{\alpha_n} = x_\beta$, which implies $x_\beta = x^*$. Therefore, for any $\alpha \in \Gamma$, there exists $n \in \mathbb{N}$ such that $x_{\alpha_n} \prec x_\alpha$, i.e. x^* is a lower bound of $(x_\alpha)_{\alpha\in\Gamma}$ (note that $x^* \prec x_{\alpha_n}$). Zorn's lemma will therefore imply that $(M, \prec)$ has minimal elements.

Now we have the following generalized Caristi's fixed point theorem.

Corollary. Let (X, d) be a complete cone metric space and $\varphi : X \to Y$ be a P-lower

semicontinuous and lower bounded map. Let $T : X \to X$ be a map such that

$$\mathbf{d(x, Tx) \leq \varphi(x) - \varphi(Tx)}, \quad \textit{for all} \quad \mathbf{x \in X}$$

(i.e. $Tx \prec x$). **Then T has a fixed point.**

Proof. By the above Lemma, the ordered set $(M, \prec)$ has a minimal element, say x^*. By our assumption $Tx^* \prec x^*$. Then by the minimality of x^*, we get $Tx^* = x^*$.

The proof of the above corollary yields the following endpoint result.

Corollary. Let (X, d) be a complete cone metric space and $\varphi : X \to Y$ be a P-lower semicontinuous and lower bounded map. Let $\mathcal{T} : X \multimap X$ be a set-valued map such that $\mathcal{T}x$ is nonempty for each $x \in X$. If the condition

$$d(x, y) \leq \varphi(x) - \varphi(y), \quad \textbf{for all} \quad y \in \mathcal{T}x$$

is satisfied (i.e. $y \prec x$), then T has a end-point in X, that is, there exists $x^* \in X$ such that $\mathcal{T}(x^*) = \{x^*\}$.

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