FIXED POINT THEOREMS FOR SET-VALUED MAPPINGS

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In this talk, we introduce the class of KKM-type mappings on metric spaces and establish some fixed point theorems for this classes. We also introduce a new generalized set-valued contraction on topological spaces with respect to a measure of noncompactness. Then we establish some fixed point theorems for the KKM-type mappings in metric spaces which are either generalized set-valued contraction or condensing. Moreover, Some applications of these results are given.

In nonlinear functional analysis, there are tree basic types of topological fixed point theorems. The first one is the so-called Fan-Browder fixed point theorem which says that a set-valued self-mapping defined in a compact convex Hausdorff topological vector spaces has at least one fixed point if the set-valued mapping has open inverse values. Browder [3, 4] show that any multimap with convex values and open inverse values from a Hausdorff compact space to a convex space has a continuous selection and used this fact to prove the Fan-Browder fixed point theorem [3, 4]. The second type is the so-called Fan-Glicksberg fixed point theorem (for example, see Fan [9] or Glicksberg [10] which says that an upper semicontinuous set-valued self-mapping defined in a compact convex subset of a Hausdorff locally convex topological vector space has at least one fixed point. The celebrated Fan-Glicksberg fixed point theorem is so general since it includes many fixed point theorems such as Kakutani fixed point theorem for upper semicontinuous set-valued in Euclidean spaces, the Tychonoff, Schauder, Bohnenlus-Karlin and Brouwer, and many other fixed point theorems for continuous (single-valued) mapping in locally convex topological vector spaces, normed spaces, Banach spaces and Euclidean spaces as special cases. The third type is the so-called Himmelberg fixed point theorem (see [11] which says that every compact upper semicontinuous multivalued map T with nonempty closed convex values from a nonempty convex subset X of a locally convex topological space E into itself has a fixed point. The Himmlberg fixed point theorem include the Fan-Glicksberg fixed point theorem.

Set-valued mappings and fixed points

The study of fixed point theorems for multi-valued mappings was initiated by Kakutani, in 1941, in finite dimensional spaces and was extended to infinite dimensional Banach spaces by Bohnenblust and Karlin, in 1950, and to locally convex spaces by Ky Fan, in 1952.

Let X and Y be two topological Hausdorff spaces and $F : X \multimap Y$ be a multifunction with nonempty values, then F is said to be:

(i) upper semi continuous (u.s.c.), if for each closed set $B \subset Y$, $F^{-}(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$ is closed in X.

(ii)lower semi continuous (l.s.c.), if for each open set $B \subset Y$, $F^{-}(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$ is open in X.

(iii) continuous , if it is both u.s.c. and l.s.c..

(iv) closed if its graph $G_r(F) = \{(x, y) \in X \times Y : y \in F(x)\}$ is closed.

(v) compact, if $clF(X)$ is a compact subset of Y.

It is well known that if Y is a compact space and T is closed, then T is u.s.c..
For a set X, we denote the set of all nonempty finite subset of X by $\langle X \rangle$.
A multifunction $F : X \multimap X$ is said to have fixed point if $x_0 \in F(x_0)$ for some $x_0 \in X$.

Theorem. (Brouwer 1912) Let B be the unit ball in $\mathbf{R}^n$ and $f : B \to B$ a continuous function. Then f has a fixed point i.e. $(\exists x \in B : f(x) = x)$.

Kakutani proved a generalization of Brouwer's theorem to set-valued mappings.

Theorem. (Kakutani(1941)) If K is a nonempty closed bounded convex subset of $\mathbf{R}^n$ and $F : K \multimap K$ is an upper semicontinuous set-valued mapping with nonempty closed convex values, then F has a fixed point.

Kakutani(1943) produced an example that Brouwer's theorem dose not hold, in general, for infinite dimensional spaces.

Example. Let $B = \{x \in l^2 :\| x \| \leq 1\}$. Define a map $f : B \to B$ by

$$f(x) = \{\sqrt{1 - \| x \|^2}, x_1, x_2, ...x_n, ...\}.$$

Then f is continuous and $\| f(x) \| = 1$.
If $x_0 = \{x_1, ..., x_n, ...\} \in B$ and $f(x_0) = x_0$, then $\| f(x_0) \| = \| x_0 \| = 1$. But

$$\begin{aligned} f(x_0) &= \{\sqrt{1 - \| x_0 \|^2}, x_1, x_2, ...x_n, ...\} \\ &= \{0, x_1, ..., x_n, ...\} \\ &= x_0 \\ &= \{x_1, .., x_n, ...\}. \end{aligned}$$

This gives $x_1 = 0, x_2 = 0, ..., x_n = 0, ...$ or $x_0 = \{0, 0, ..., 0, ...\}$.

Dugundji (1951) proved the following theorem:

Theorem. A closed unit ball in a normed linear space is a fixed point space if and only if it is compact.

Klee (1955) generalized this result to arbitrary convex sets in metrizable locally convex spaces.

Brouwer's theorem was extended to infinite dimensional spaces by Schauder in 1930 in the following way.

Theorem. (Schauder(1930)) Let X be a Banach space, C compact convex subset of X and $f : C \to C$ is a continuous map. Then f has at least one fixed point in C.

The multivalued analogue of Schauder's fixed point theorem was given by Bohnenblust and Karlin.

Theorem. (Bohnenblust-Karlin(1950)) If K is a nonempty compact convex subset of a Banach space X and $F : K \multimap K$ is an upper semicontinuous multifunction with nonempty closed convex values, then F has a fixed point.

In 1935, Tychonoff extended Brouwer's result to a compact convex subset of a locally convex space.

Theorem. (Tychonoff(1935)) Let C be a nonempty compact convex subset of a locally convex space X and $f : C \to C$ is a continuous map. Then f has a fixed point.

The multivalued analogue of Tychonoff's fixed point theorem was given by Fan and Glicksberg independently.

Theorem. (Fan-Glicksberg(1952)) If K is a nonempty compact convex subset of a locally convex space X and $F : K \multimap K$ is an upper semicontinuous multifunction with nonempty closed convex values, then F has a fixed point.

In 1972, Himmelberg generalized the Fan-Glicksberg fixed point theorem as follows:

Theorem. Himmelberg(1972)) If K is a nonempty convex subset of a locally convex space X and $F : K \multimap K$ is an upper semicontinuous multifunction with nonempty closed convex values and compact, then F has a fixed point.

Question. Schauder) Does a compact convex subset of an arbitrary linear topological space have the fixed point property?
This problem proved to be very difficult and for over 65 years defied the effort of many mathematiciants. An affirmative answer was given only recently by Cauty (2001).

KKM MAPPINGS IN METRIC SPACES

Suppose that A is a bounded subset of a metric space (M, d). Then:
(i) $co(A) = \bigcap\{B \subseteq M : B$ is a closed ball in M , $A \subset B\}$.
(ii) $\mathcal{A}(M) = \{A \subseteq M : A = co(A)\}$ i.e. $A \in \mathcal{A}(M)$ if and only if A is an intersection of closed balls containing A. In this case, we say that A is admissible subset of M.
(iii) A is called subadmissible, if for each $D \in \langle A\rangle$, $co(D) \subseteq A$. Obviously, if A is an admissible subset of M, then A must be subadmissible.

Let (M, d) be a metric space, X a nonempty subset of M. A multifunction $G : X \multimap M$ is called a KKM mapping, if for each $A \in \langle X \rangle$, $co(A) \cap X \subset G(A)$. More generally if Y is a topological space and $G : X \multimap Y$, $F : X \multimap Y$ are two multifunctions such that for any $A \in \langle X \rangle$, $F(co(A) \cap X) \subseteq G(A)$, then G is called a generalized KKM mapping with respect to F. If the multifunction $F : X \multimap Y$ satisfies the requirement that for any generalized KKM mapping $G : X \multimap Y$ with respect to F the family $\{clG(x) : x \in X\}$ has the finite intersection property, then F is said to have the KKM property. We define

$KKM(X,Y) := \{F : X \multimap Y : F$ has the KKM prope

When X is convex subset of topological vector space, the class $KKM(X,Y)$ was introduced and studied by Chang and Yen [5].

Horvath [12] found that hyperconvex spaces are a particular type of C-spaces, hence they are G-convex spaces [18] . Fakhar and Zafarani [8, Lemma 2.7] have shown that those multifunctions defined on G-convex spaces which are closed, compact and acyclic valued have the KKM property.

Let X be a nonempty subset of a metric space M. Then $F : X \multimap M$ is said to have an approximate fixed point if for any $\varepsilon > 0$, there exists an $x_\varepsilon \in X$ such that $F(x_\varepsilon) \cap B(x_\varepsilon, \varepsilon) \neq \emptyset$.

Theorem.(AFZ[2]2005) Let (M, d) be a metric space and X a nonempty subadmissible subset of M. Suppose that $F \in KKM(X, X)$ such that $clF(X)$ is totally bounded, then F has an approximate fixed point.

Theorem. (AFZ[2]2005) Let (M, d) be a metric space and X a nonempty subadmissible subset of M. Suppose that $F \in KKM(X, X)$ is closed and compact, then F has a fixed point.

Corollary. Let (M, d) be a metric space and X a nonempty subadmissible subset of M. Suppose that the identity mapping $I : X \to X$ belongs to $KKM(X, X)$, then any continuous mapping $f : X \to X$ such that $clf(X)$ is compact, has a fixed point.

Remark. The KKM principle of Fan implies that the identity mapping in normed spaces is an element of $KKM(X, X)$ for any convex set X. Khamsi [14], has shown that when M is hyperconvex, then $I \in KKM(X, X)$ for $X \in \mathcal{A}(M)$. This result is also true for a metric topological vector space E such that all balls are convex. In fact by Fan's KKM principle, the identity mapping belongs to $KKM(X, X)$ for each convex subset X of E. Hence the identity mapping also belongs to $KKM(X, X)$ with respect to metric of E for any admissible subset X of E.

Here we obtain a generalized Fan's matching theorem for metric spaces. In fact we establish an open version of Fan's matching theorem which improves Theorem 2.7 of Yuan [20] and is similar to Theorem 4.4 of Chang and Yen [5]. Let us recall that a subset A of a topological space Z is called compactly open, if its intersection with any compact subsets of Z is open in its relative topology.

Theorem. Let $X \in \mathcal{A}(M)$ of a metric space (M, d) and Z a topological space. Suppose that $F \in KKM(X, Z)$ is compact and $T : X \multimap Z$ is compactly open valued such that $clF(X)$ is contained in $T(X)$. Then there exists $\{x_1, ..., x_j\} \subset X$ such that:

$$F(co(\{x_1, ..., x_j\}) \cap \bigcap_{k=0}^{j} T(x_k) \neq \emptyset.$$

As an application of the above theorem, we have the following form of the Fan-Browder type fixed point theorem, see Kirk et al. [13, Theorem 3.1].

Corollary. Let $X \in \mathcal{A}(M)$ be a compact subset of a metric space (M, d) such that the identity mapping $I \in KKM(X, X)$. Suppose that $G : X \multimap X$ is a multifunction with admissible values such that $X = \bigcup\{IntG^{-}(y) : y \in X\}$. Then G has a fixed point.

FIXED POINT THEOREMS FOR GENERALIZED SET-CONTRACTION

In 1930 Kuratowski [15] introduced a measure of noncompactness α of bounded sets in a metric space, in order to generalize the Cantor intersection theorem. Let (M, d) be a metric space and $B \subseteq M$ be a bounded set, then

$$\alpha(B) = \inf\{\delta > 0 : B \subseteq \cup_{i=1}^{n} A_i,\ \operatorname{diam}(A_i) < \delta\}.$$

Definition. *Let X be a nonempty subset of M and $f : X \to X$, then*

- *f is said to be an α-k-set contraction if f is bounded and there is a $k \in [0, 1)$ such that $\alpha(fB) \leq k\alpha(B)$ for all bounded subsets B of X.*

- *f is said to be condensing if f is bounded and if $\alpha(fB) < \alpha(B)$ for all bounded subsets B of X for which $\alpha(B) > 0$.*

Darbo [6] showed that if X is a closed, bounded and convex subset of a Banach space and $f : X \to X$ is a continuous α-k-set contraction, then f has a fixed point. Later, Sadovskii [19] introduced the notion of condensing map and by transfinite induction showed that if f is a continuous condensing map, then f has a fixed point. Notice that every α-k-set contraction is condensing but the converse is not true [17]. Many authors extended the above results in different spaces; see [14, 17] and references therein.

Let E be a topological space. A measure of noncompactness is simply any functional $\mu : 2^E \to [0, \infty]$ such that:

(i) $\mu(\overline{A}) = \mu(A)$ for all $A \in 2^E$;

(ii) $\mu(A) = 0$ if and only if A is precompact;

(iii) $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$.

A sequence $\{A_n\}_{n=1}^{\infty}$ of nonempty closed subsets of E is called μ- *descending* if $A_{n+1} \subseteq A_n$ for each n and $\lim_{n\to\infty} \mu(A_n) = 0$. We say that μ has the Kuratowski property, if the intersection $A = \bigcap_{n\in\mathbb{N}} A_n$ is nonempty and compact for any μ-descending sequence $\{A_n\}_{n=1}^{\infty}$. Notice that, if E is a complete metric space, then the Hausdorff measure of noncompactness and the Kuratowski measure of noncompactness have the Kuratowski property [1].

Example. *Let E be a Banach space. Let $\mathcal{B}(E)$ and $\mathcal{WC}(E)$ denote the families of the bounded subsets of E and of the weakly compact subsets of E, respectively. The weak noncompactness measure of $B \in \mathcal{B}(E)$ [8] is defined by*

$$\beta(B) = \inf\{\epsilon > 0 : \exists \quad A \in \mathcal{WC}(E), B \subseteq A + \epsilon B_1(0)\}.$$

The weak noncompactness measure is a measure of noncompactness with the Kuratowski property [7, 16].

For some other examples of measure of noncompactness with the Kuratowski property one can refer to [1].

Definition. *Let E be a topological space and μ be a measure of noncompactness on E. Suppose that $F : E \multimap E$ is a set-valued map. Then, F is said to be*

- *μ-k set contraction, if there exists $k \in (0,1)$ such that $\mu(F(A)) \leq k\mu(A)$ for all $A \in 2^E$;*

- *generalized μ-set contraction, if for each $\epsilon > 0$, there exists $\delta > 0$ such that for $A \subseteq E$ with $\epsilon \leq \mu(A) < \epsilon + \delta$, there exists $n \in \mathbb{N}$ such that $\mu(F^n(A)) < \epsilon$.*

In the following result, we obtain the relationship between the above notions.

Proposition. Let E be a topological space and μ be a measure of noncompactness on E. Suppose that $F : E \multimap E$ is a μ-k set contraction on E, then F is a generalized μ-set contraction.

The following provides an example of a generalized μ-set contraction which is not a μ-k set contraction.

Example. *Suppose that* $M = \{1\} \bigcup \{2n, 3n : n \in \mathbb{N}\}$ *is equipped with the discrete metric. Assume that* $F : M \multimap M$ *is defined as follows:*

$$F(x) = \begin{cases} \{1\} & \text{if } x = 1, \\ \{3(2n+1)\} & \text{if } x = 2n, \\ \{1\} & \text{if } x = 3n \text{ and } n \text{ is odd.} \end{cases}$$

If a is the Kuratowski measure of noncompactness, then $\alpha(F^2(M)) = 0$. Therefore, F is a generalized α-set contraction. But if $A = \{2n : n \in \mathbb{N}\}$, then $\alpha(A) = \alpha(F(A)) = 1$ and so F is not α-k-set contraction.

Lemma. Let E be a topological space and μ be a measure of noncompactness on E. Suppose that F is a generalized μ-set contraction on E. Then, for every subset A of E which $F(A) \subseteq A$ and $\mu(A) < \infty$, we have

$$\lim \mu(F^n(A)) = 0.$$

Remark. It is easy to see that if $\lim \mu(F^n(A)) = 0$ for any subset A of E, then F is a generalized μ-set contraction on E.

Lemma. Let E be a topological space, μ be a measure of noncompactness on E with the Kuratowski property and X be a nonempty subset of E with $\mu(X) < \infty$. Suppose that $F : X \multimap X$ is a generalized μ-set contraction with nonempty compact values and $\overline{F(X)} \subseteq X$. Then, there exists a precompact subset K of X with $\overline{K} \subseteq X$ such that $F(K) \subseteq K$.

Theorem. Let (M, d) be a metric space, μ a measure of noncompactness on M with the Kuratowski property and X be a nonempty subset of M with $\mu(X) < \infty$. Suppose that $F \in KKM(X, X)$ is a closed generalized μ-set contraction, with nonempty compact values and $\overline{F(X)} \subseteq X$. Then F has a fixed point.

As an application of the above Theorem we obtain a coincidence theorem.

Theorem. Let μ be a measure of non-compactness with the Kuratowski property on the metric space M and X be a nonempty subadmissible subset of M. Suppose that $F, G : X \multimap X$ are two set-valued mappings satisfying the following conditions:

(1) $F \in KKM(X, X)$;

(2) G has nonempty subadmissible values and for every compact subset C of X and any $y \in X$, $G^{-}(y) \cap C$ is open in C;

(3) F is a generalized μ-set contraction map with compact values and $\overline{F(X)} \subseteq X$.

Then, there exists $x_0, y_0 \in X$ such that $y_0 \in F(x_0)$ and $x_0 \in G(y_0)$.

Remark.

(a) In the above theorem, instead of (3), we can assume the following condition:

[$(3)'$] G is a generalized μ-set contraction with compact values and $\overline{G(X)} \subseteq X$.

(b) If the identity mapping $I \in KKM(X, X)$ and condition $(3)'$ is satisfied, then we have a Fan-Browder fixed point theorem for the set-valued map G.

As an application of the above Theorem, we deduce the existence of a fixed point for contractive mappings. Recall that a mapping $f : X \to X$, where X is subset of a Banach space $(E, \|\|)$, is called contractive if, $\|f(x) - f(y)\| < \|x - y\|$ for each $x \neq y \in X$.

Theorem. Let X be a weakly closed bounded subset of a Banach space E and β be the weak measure of noncompactness on E. Suppose that $f : X \to X$ is a contractive generalized β-set contraction and weakly continuous. Then f has a fixed point.

Lemma. Let E be a topological space, μ be a measure of noncompactness on E and X be a nonempty closed subset of E with $\mu(X) < \infty$. Let $F : X \multimap X$ be a μ-condensing set-valued map. Then, there exists a compact subset K of X such that $F(K) \subseteq K$.

Theorem. Let (M, d) be a metric space, μ be a measure of noncompactness on M and X be a nonempty closed subset of M with $\mu(X) < \infty$. If $F \in KKM(X, X)$ is a closed μ-condensing set-valued map, then F has a fixed point.

THANK YOU

References

[1] R. R. Akhmerov, M. I. Kamenskiĭ, A. S. Potapov, A. E. Rodkina, B. N. Sadovskiĭ, Measures of noncompactness and condensing operators, Birkh?user Verlag, Basel, 1992.

[2] A. Amini, M. Fakhar and J. Zafarani, KKM mappings in metric spaces, 60 (2005), no. 6, 1045–1052.

[3] F.E. Browder, A new generalization of the Shauder fixed point theorem, Math. Ann. 174 (1967), 285-390.

[4] F. E. Browder, The fixed point theory of multivalued mappings in topological vector spaces, Math. Ann. 177(1968), 283-301.

[5] T. H. Chang and C. L. Yen, KKM property and fixed point theorems, J. Math. Anal. Appl. 203 (1996), 224-235.

[6] G. Darbo, Punti uniti in trasformazioni a codominio non compatto, Rend. Sem. Mat. Univ. Padova 24, (1955),

[7] F. De Blasi, the measure the weak non compactness of the unit sphere in a Banach space is either zero or one, Ist. Mat. Ulissebini, 7, 1974/75, Firenze.

[8] M. Fakhar and J. Zafarani, Fixed points theorems and quasi-variational inequalities in G-convex spaces, Bull. Belg. Math. Soc. 12 (2005), no. 2, 235–247

[9] K. Fan, Fixed-point and minimax theorems in locally convex topological linear spaces.

Proc. Nat. Acad. Sci. U. S. A. 38, (1952). 121–126.

[10] I. L. Glicksberg, A further generalization of the Kakutani fixed theorem, with application to Nash equilibrium points. Proc. Amer. Math. Soc. 3, (1952). 170–174.

[11] C. J. Himmelberg, Fixed points of compact multifunctions. J. Math. Anal. Appl. 38 (1972), 205–207.

[12] C. Horvath, Extension and selection theorems in topologiacl spaces with a generalized convexity structure, Ann. Fac. Sci. Toulouse Math. 2 (1993), 253-269.

[13] W. E. Kirk, B. Sims and X. Z. Yuan, The Knaster-Kuratowski and Mazurkiewicz theory in hyperconvex metric spaces and some

of its applications, Nonlinear Anal. T.M.A. 39 (2000), 611-627.

[14] M. A. Khamsi, KKM and Ky Fan theorems in hyperconvex spaces, J. Math. Anal. Appl. 204 (1996), 298-306.

[15] C. Kuratowski, Sur les espaces completes, Fund. Math. **15** (1930), 301-309.

[16] A. J. B. Lopes-Pinto, Fixed point theorems for β-contractions, Centro Mat. Fund., **19**, 1979, Lisboa.

[17] R. D. Nussbaum, The fixed point index for local condensing maps, Ann. Mat. Pura Appl. **89(4)** (1971), 217-258.

[18] S. Park, Fixed point of better admissible maps on generalized convex spaces, J. Korean Math. Soc. **37** (2000), 885-899.

[19] B. N. Sadovskii, On fixed point principle, Funktsional. Analis, **4** (1967), 74-76.

[20] G. X. Z. Yuan, The characterization of generalized metric KKM mappings with open values in hyperconvex metric spaces and some applications, J. Math. Anal. Appl. 235 (1999), 315-325.