Three-Dimensional Manifolds and Heegaard Floer Homology

Eaman Eftekhary

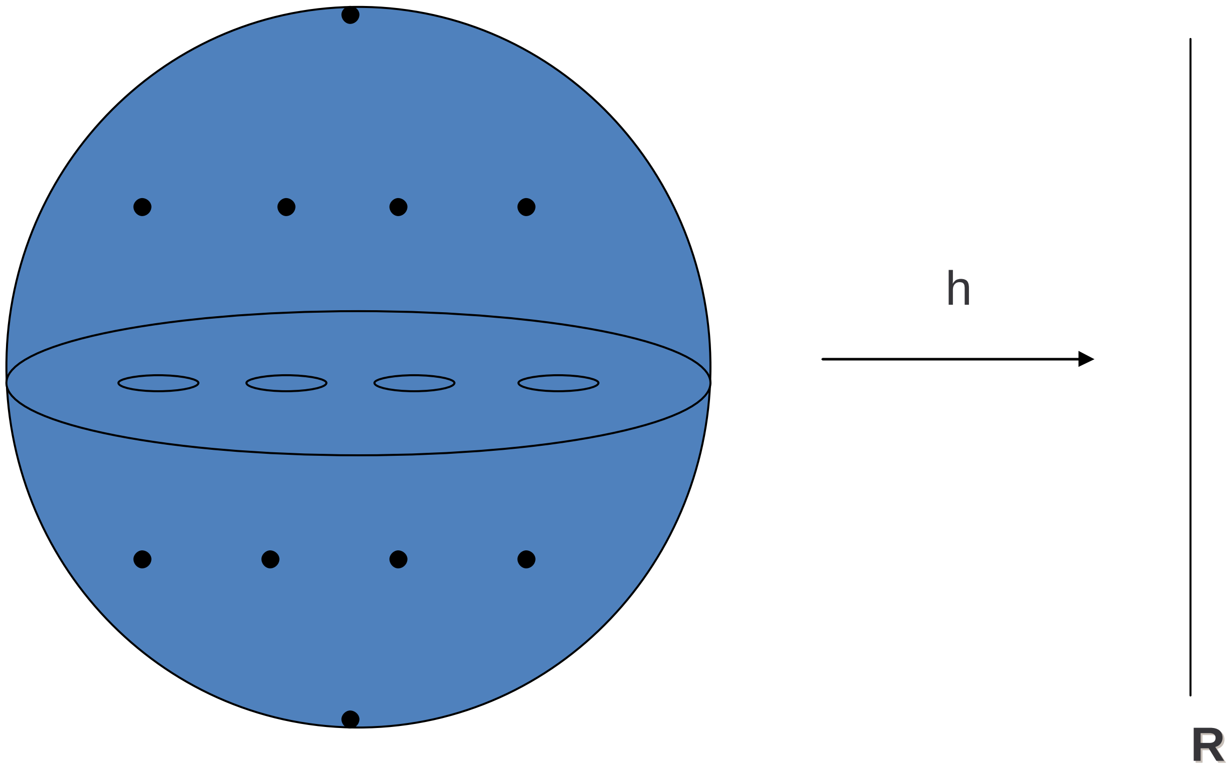
IPM, Tehran

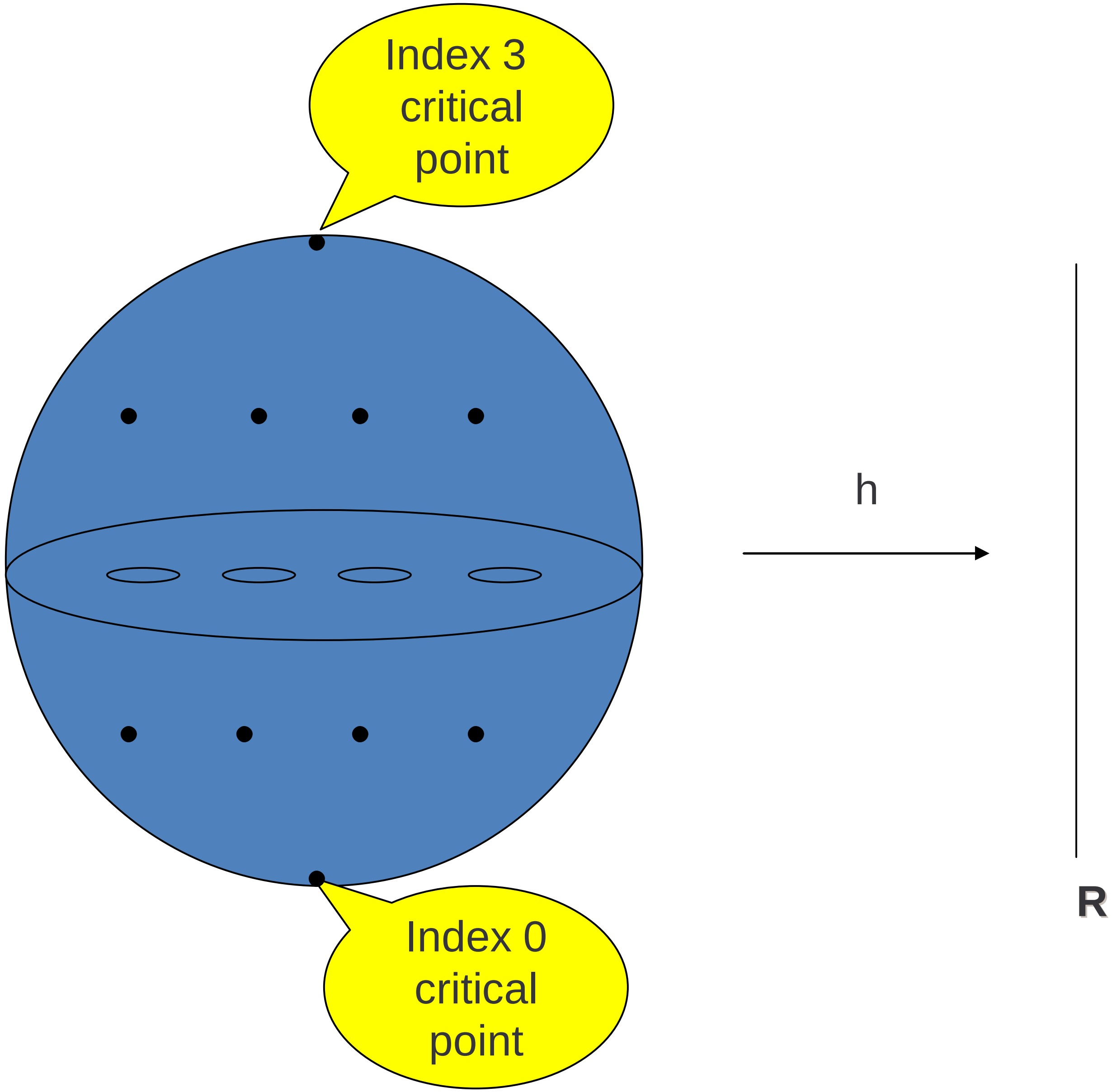
General Construction

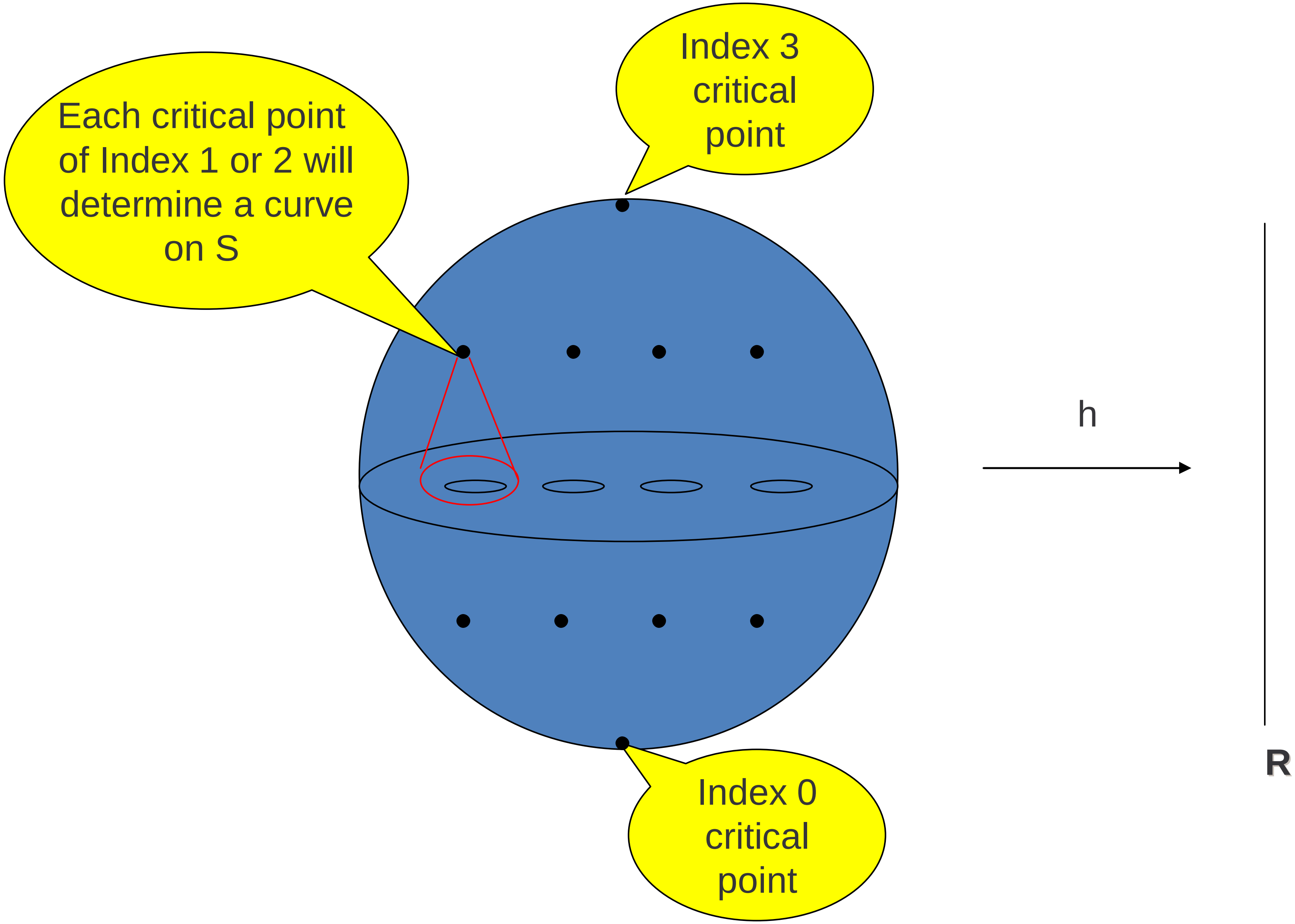
- Suppose that Y is a compact oriented three-manifold equipped with a self-indexing Morse function h with a unique minimum, a unique maximum, g critical points of index 1 and g critical points of index 2.
- The pre-image of 1.5 under h will be a surface of genus g which we denote by S.

General Construction

- Suppose that Y is a compact oriented three-manifold equipped with a self-indexing Morse function h with a unique minimum, a unique maximum, g critical points of index 1 and g critical points of index 2.


h
R


Index 3 critical point
h
R
Index 0 critical point


Each critical point of Index 1 or 2 will determine a curve on S
Index 3 critical point
h
Index 0 critical point
R

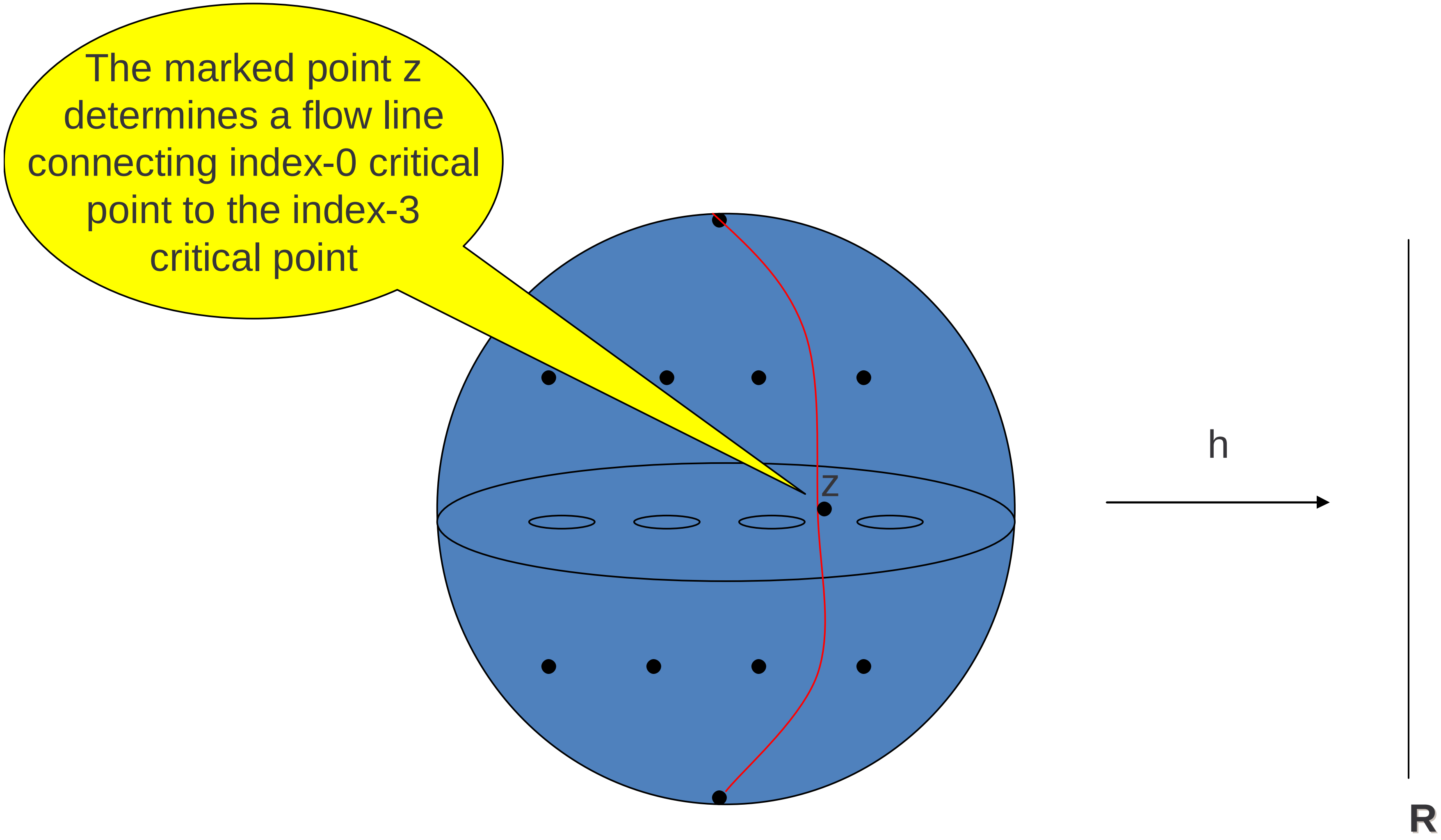
Heegaard diagrams for three-manifolds

- Each critical point of index 1 or 2 determines a simple closed curve on the surface S. Denote the curves corresponding to the index 1 critical points by α_i, i=1,…,g and denote the curves corresponding to the index 2 critical points by β_i, i=1,…,g.

Heegaard diagrams for three-manifolds

- Each critical point of index 1 or 2 determines a simple closed curve on the surface S. Denote the curves corresponding to the index 1 critical points by α_i, i=1,…,g and denote the curves corresponding to the index 2 critical points by β_i, i=1,…,g.
- The curves α_i, i=1,…,g are (homologically) linearly independent. The same is true for β_i, i=1,…,g.

- We add a marked point z to the diagram, placed in the complement of these curves. Think of it as a flow line for the Morse function h, which connects the index 3 critical point to the index 0 critical point.



The marked point z determines a flow line connecting index-0 critical point to the index-3 critical point
z
h
R

- We add a marked point z to the diagram, placed in the complement of these curves. Think of it as a flow line for the Morse function h, which connects the index 3 critical point to the index 0 critical point.
- The set of data

$$H=(S, (\alpha_1,\alpha_2,\ldots,\alpha_g),(\beta_1,\beta_2,\ldots,\beta_g),z)$$

is called a pointed Heegaard diagram for the three-manifold Y.

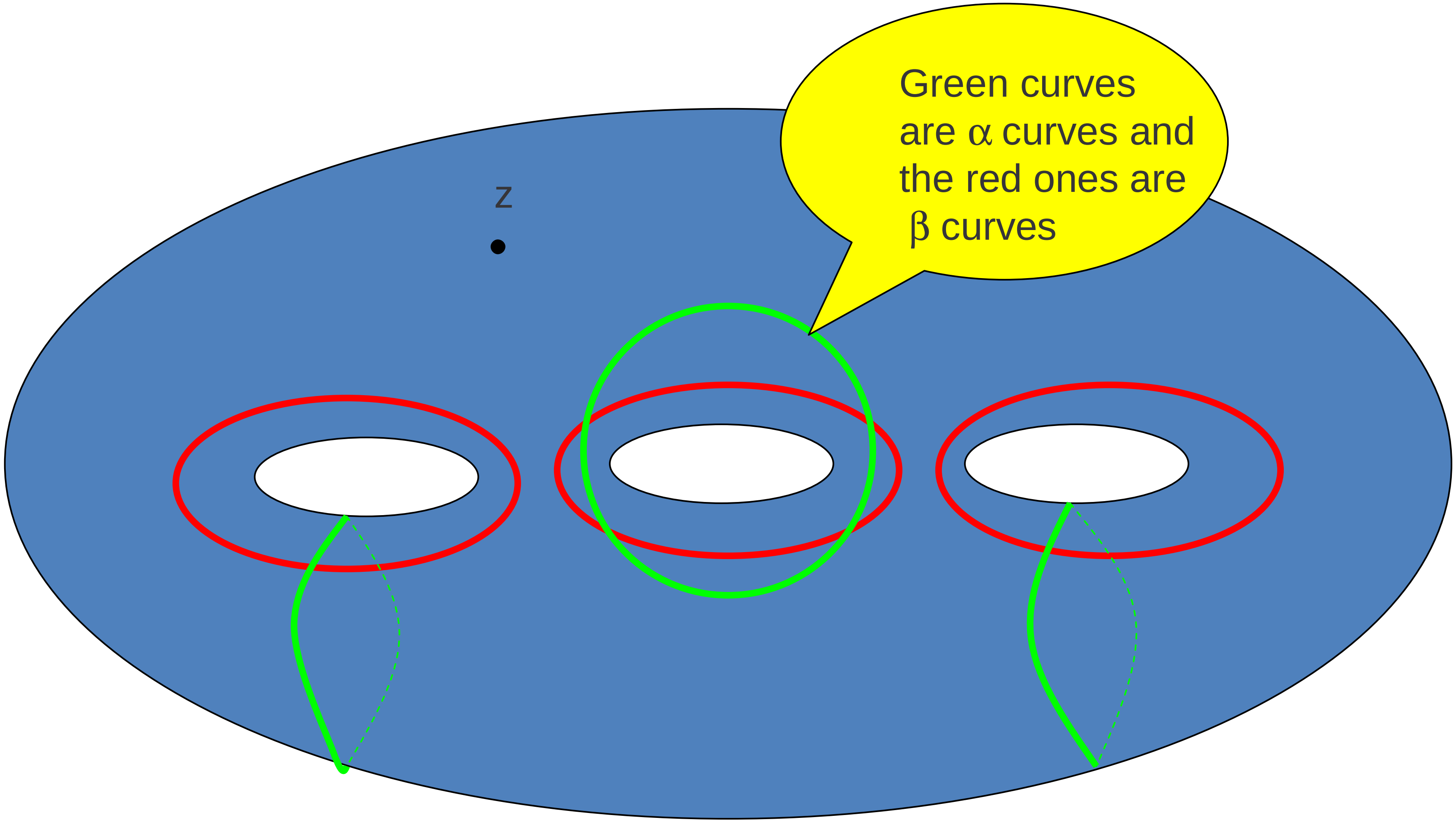
- We add a marked point z to the diagram, placed in the complement of these curves. Think of it as a flow line for the Morse function h, which connects the index 3 critical point to the index 0 critical point.
- The set of data

$$H=(S, (\alpha_1,\alpha_2,\dots,\alpha_g),(\beta_1,\beta_2,\dots,\beta_g),z)$$

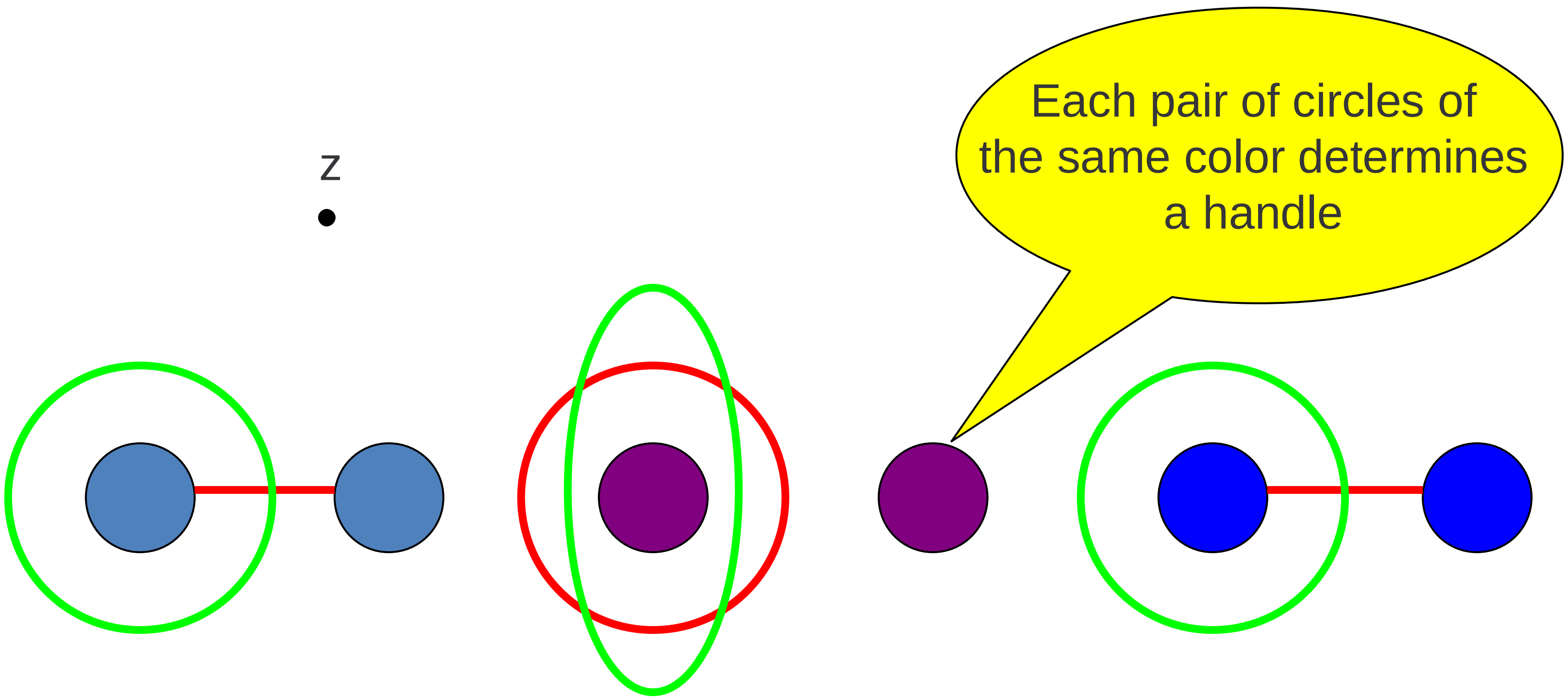
is called a pointed Heegaard diagram for the three-manifold Y.

- H uniquely determines the three-manifold Y but not vice-versa

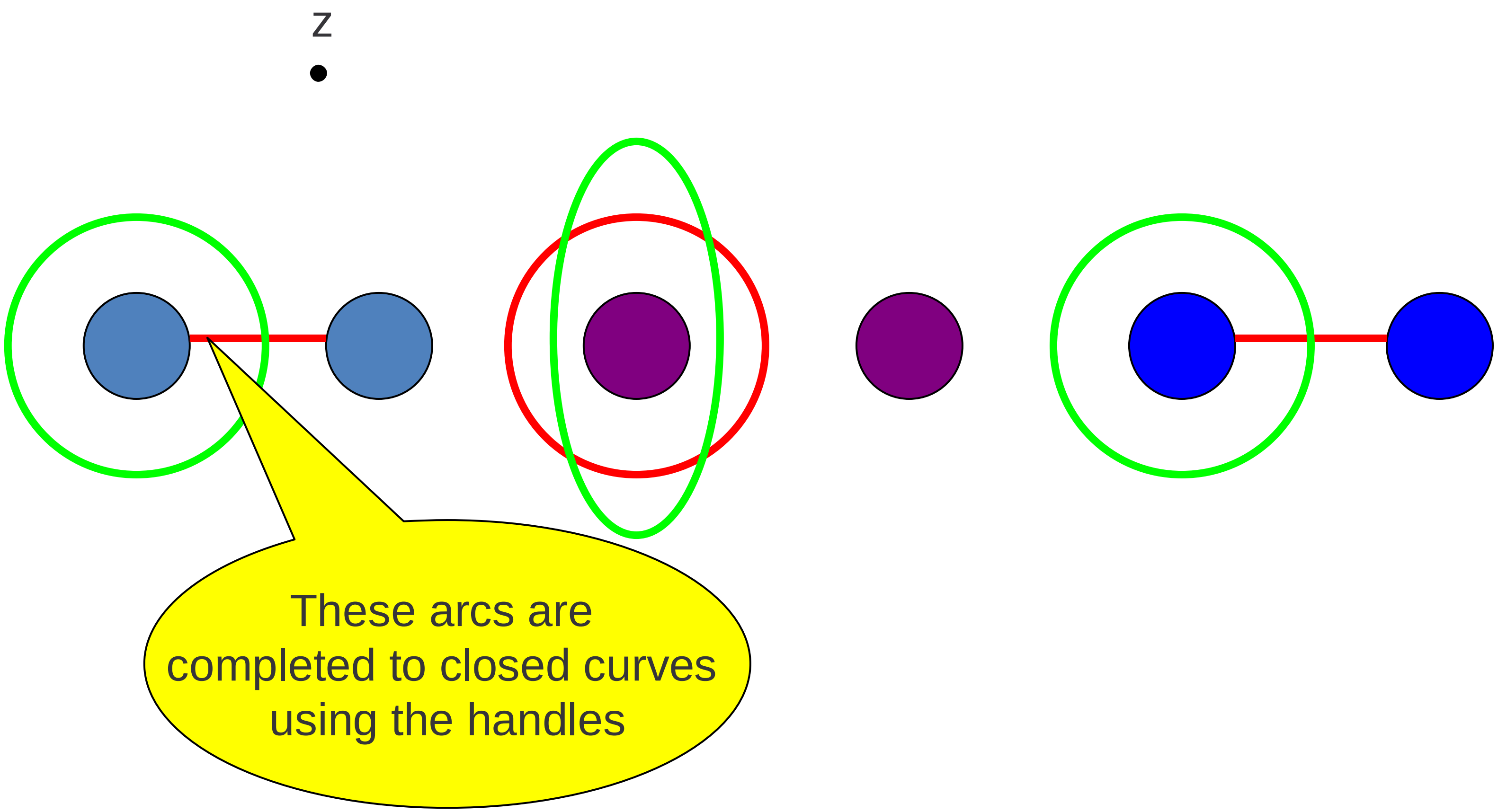
A Heegaard Diagram for $S^1 \times S^2$



A different way of presenting this Heegaard diagram



A different way of presenting this Heegaard diagram



Knots in three-dimensional manifolds

- Any map embedding S^1 to a three-manifold Y determines a homology class $\beta \in H_1(Y,\mathbf{Z})$.

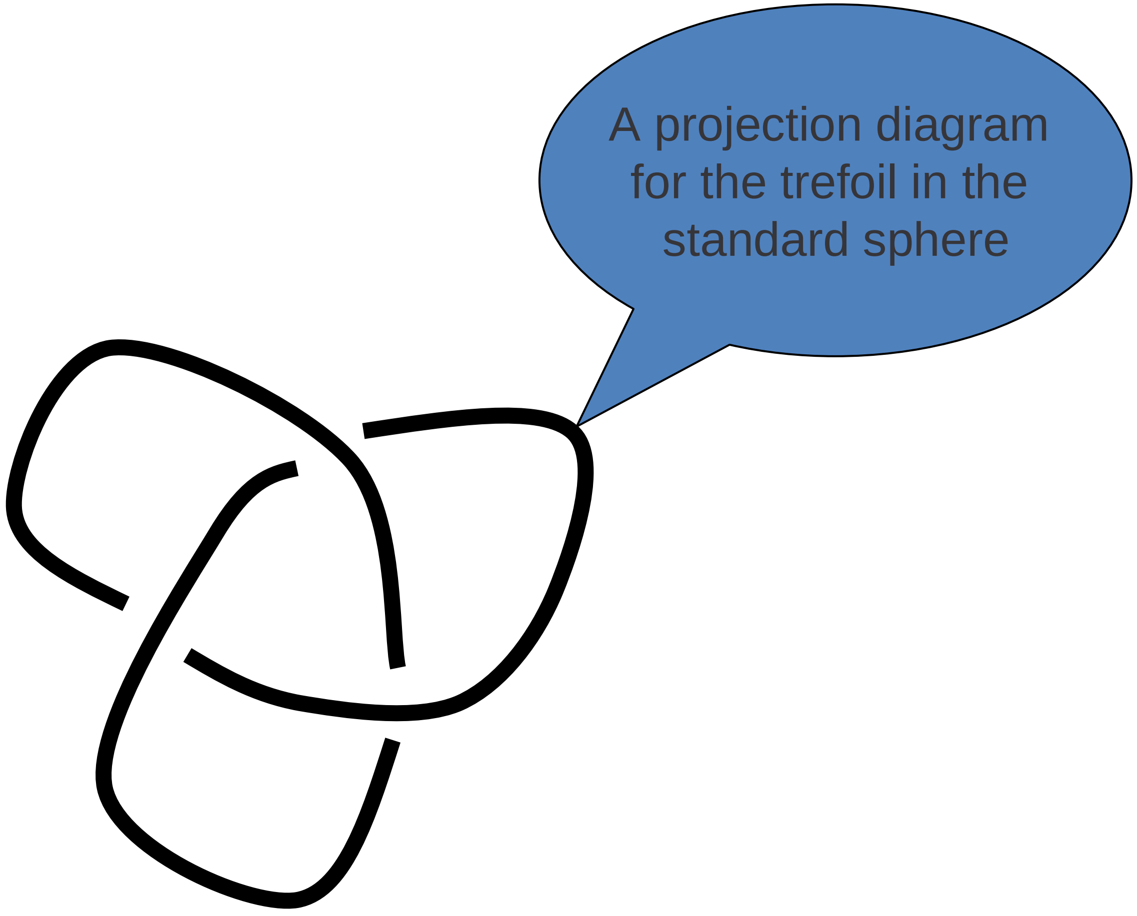
Knots in three-dimensional manifolds

- Any map embedding S^1 to a three-manifold Y determines a homology class $\beta \in H_1(Y,\mathbf{Z})$.
- Any such map which represents the trivial homology class is called a knot.

Knots in three-dimensional manifolds

- Any map embedding S^1 to a three-manifold Y determines a homology class $\beta \in H_1(Y,\mathbf{Z})$.
- Any such map which represents the trivial homology class is called a knot.
- In particular, if $Y=S^3$, any embedding of S^1 in S^3 will be a knot, since the first homology of S^3 is trivial.

Trefoil in S^3

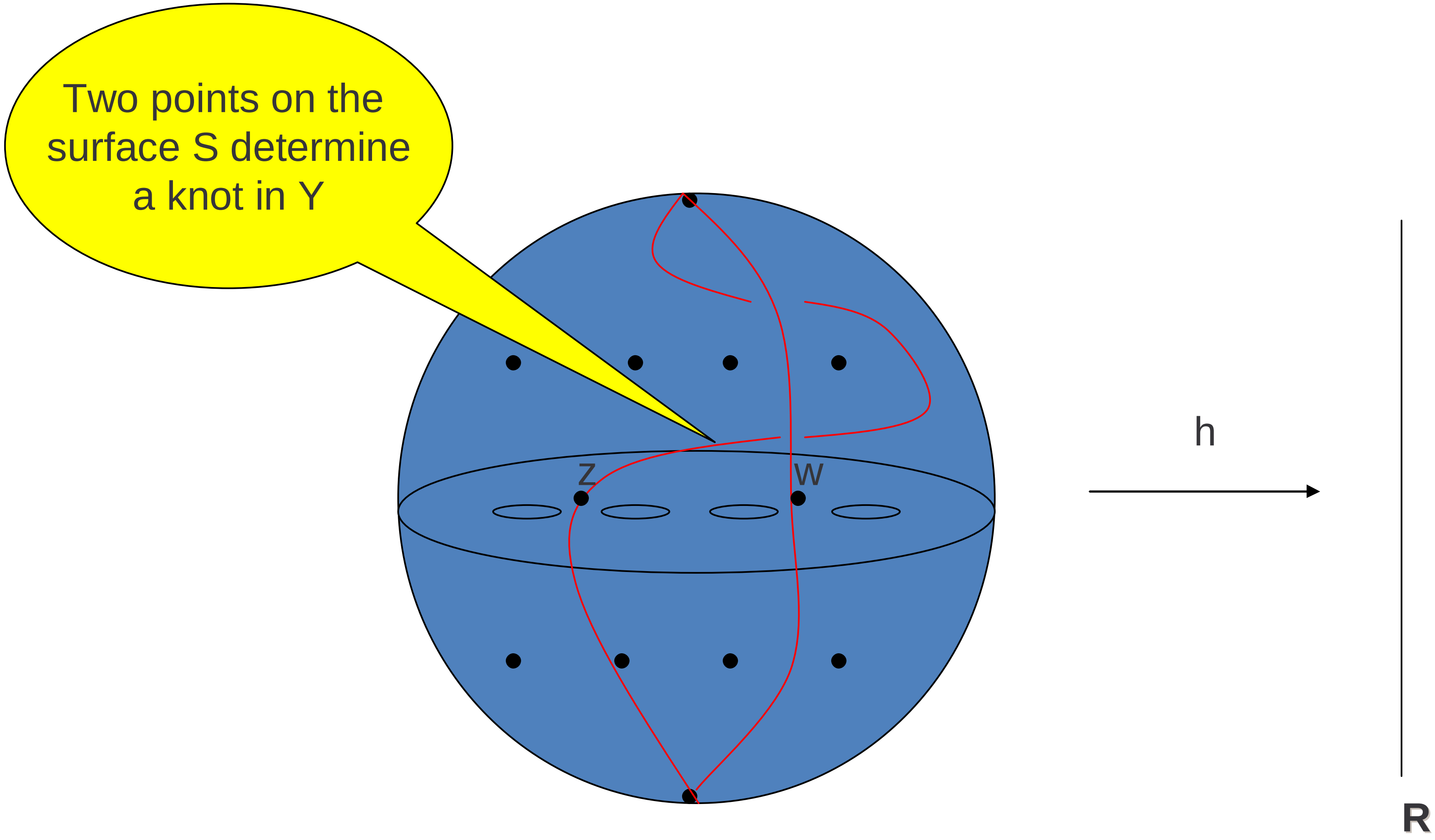


Heegaard diagrams for knots

- A pair of marked points on the surface S of a Heegaard diagram H for a three-manifold Y determine a pair of paths between the critical points of indices 0 and 3. These two arcs together determine an image of S^1 embedded in Y.

Heegaard diagrams for knots

- A pair of marked points on the surface S of a Heegaard diagram H for a three-manifold Y determine a pair of paths between the critical points of indices 0 and 3. These two arcs together determine an image of S^1 embedded in Y.
- Any knot in Y may be realized in this way using some Morse function and the corresponding Heegaard diagram.



Two points on the surface S determine a knot in Y
Z
W
h
R

Heegaard diagrams for knots

- A Heegaard diagram for a knot K is a set $H=(S, (\alpha_1,\alpha_2,\ldots,\alpha_g),(\beta_1,\beta_2,\ldots,\beta_g),z,w)$

  where z,w are two marked points in the complement of the curves $\alpha_1,\alpha_2,\ldots,\alpha_g$, and $\beta_1,\beta_2,\ldots,\beta_g$ on the surface S.

Heegaard diagrams for knots

- A Heegaard diagram for a knot K is a set $H=(S, (\alpha_1,\alpha_2,\ldots,\alpha_g),(\beta_1,\beta_2,\ldots,\beta_g),z,w)$ where z,w are two marked points in the complement of the curves $\alpha_1,\alpha_2,\ldots,\alpha_g$, and $\beta_1,\beta_2,\ldots,\beta_g$ on the surface S.
- There is an arc connecting z to w in the complement of $(\alpha_1,\alpha_2,\ldots,\alpha_g)$, and another arc connecting them in the complement of $(\beta_1,\beta_2,\ldots,\beta_g)$. Denote them by ε_α and ε_β.

Heegaard diagrams for knots

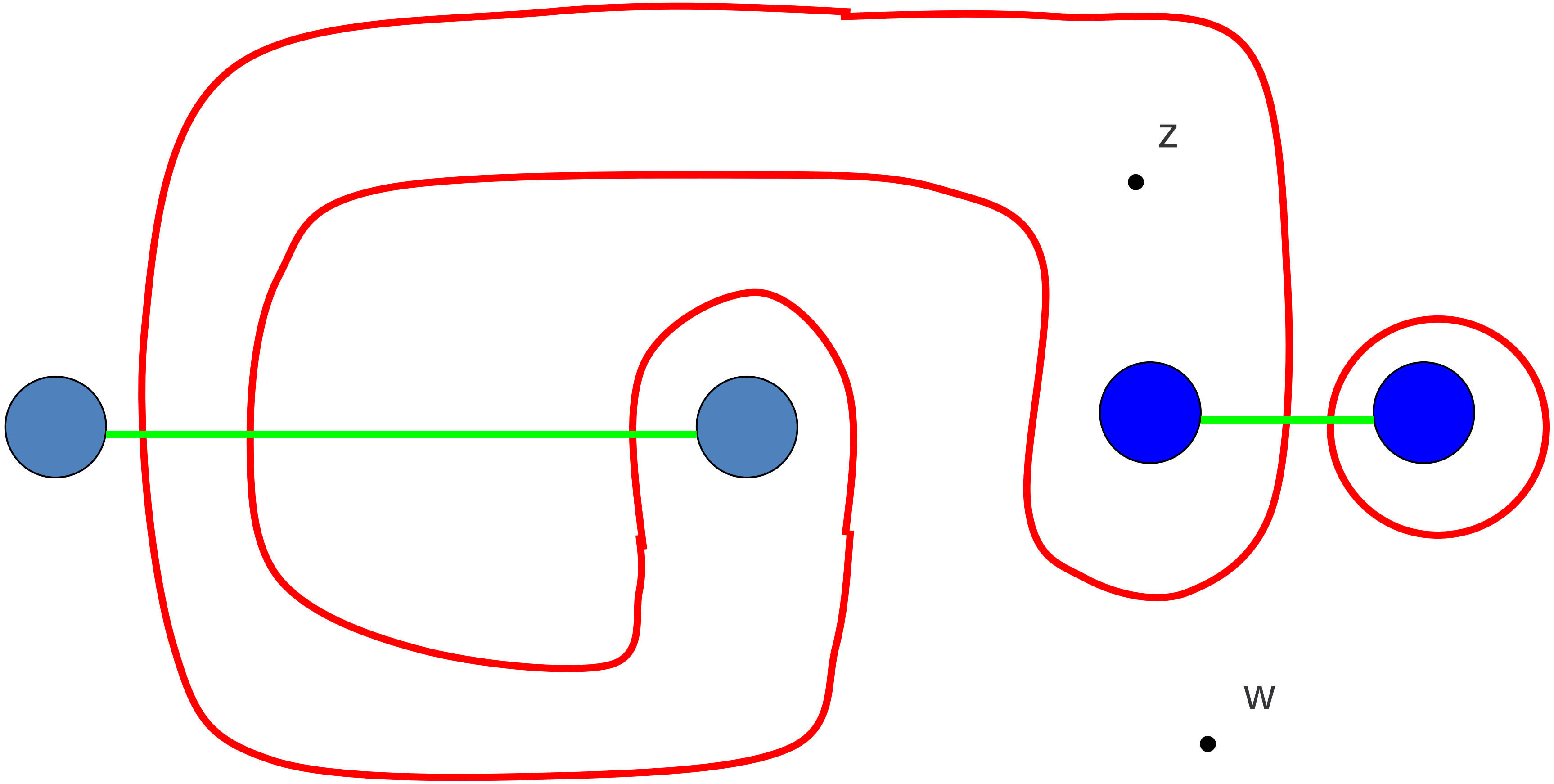
- The two marked points z,w determine the trivial homology class if and only if the closed curve $\varepsilon_\alpha - \varepsilon_\beta$ can be written as a linear combination of the curves $(\alpha_1,\alpha_2,\ldots,\alpha_g)$, and $(\beta_1,\beta_2,\ldots,\beta_g)$ in the first homology of S.

Heegaard diagrams for knots

- The two marked points z,w determine the trivial homology class if and only if the closed curve ε_α-ε_β can be written as a linear combination of the curves $(\alpha_1,\alpha_2,\ldots,\alpha_g)$, and $(\beta_1,\beta_2,\ldots,\beta_g)$ in the first homology of S.
- The first homology group of Y may be determined from the Heegaard diagram H:

$$H_1(Y,\mathbf{Z})=H_1(S,\mathbf{Z})/[\alpha_1=\ldots=\alpha_g=\beta_1=\ldots=\beta_g=0]$$

A Heegaard diagram for the trefoil



Constructing Heegaard diagrams for knots in S^3

- Consider a plane projection of a knot K in S^3.

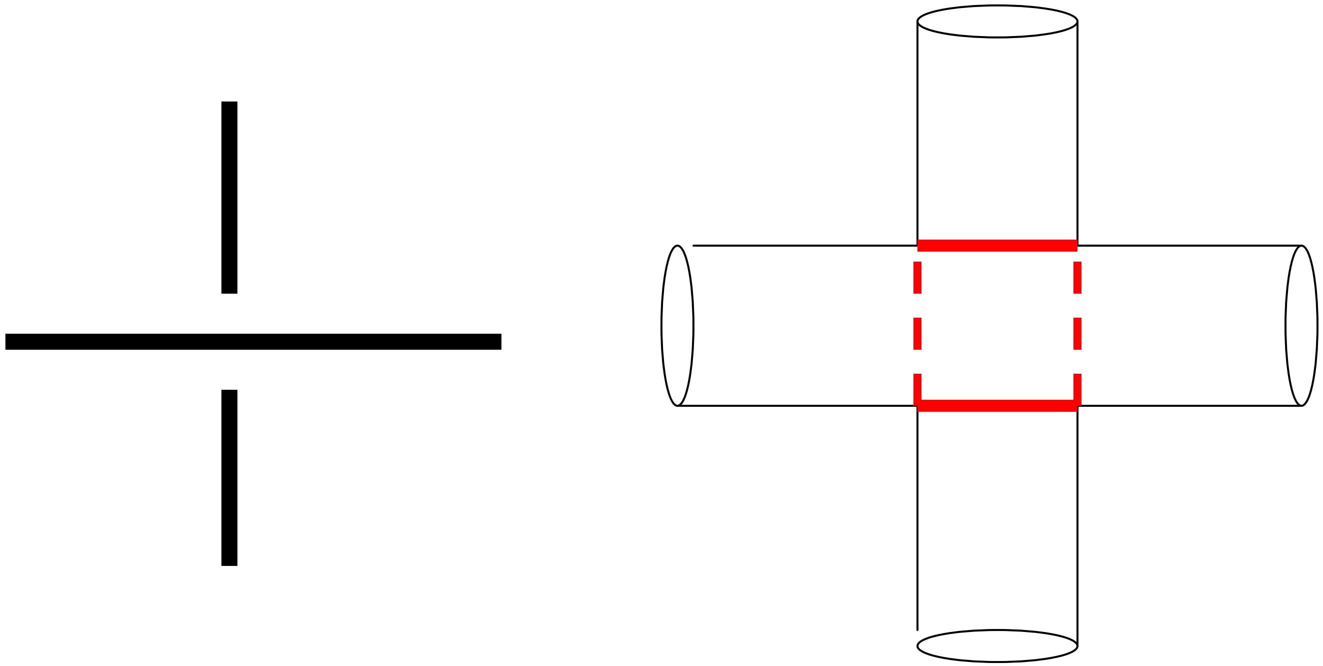
Constructing Heegaard diagrams for knots in S^3

- Consider a plane projection of a knot K in S^3.
- Construct a surface S by thickening this projection.

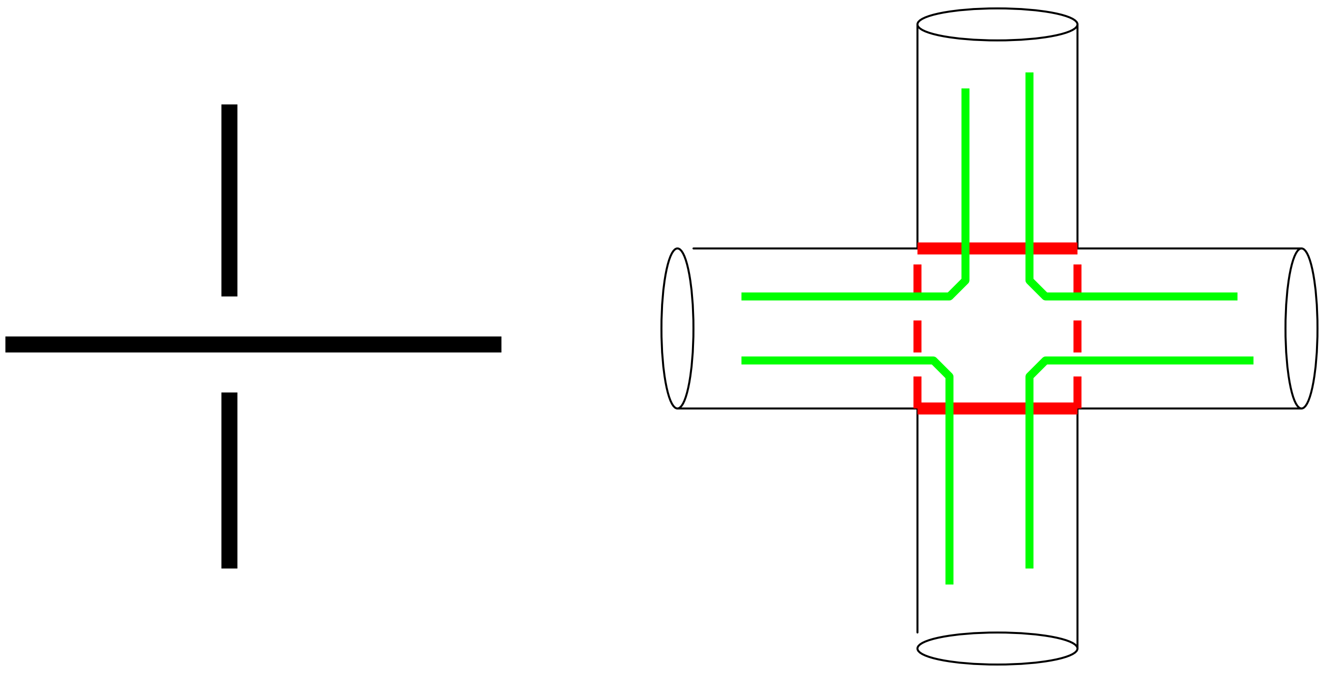
Constructing Heegaard diagrams for knots in S^3

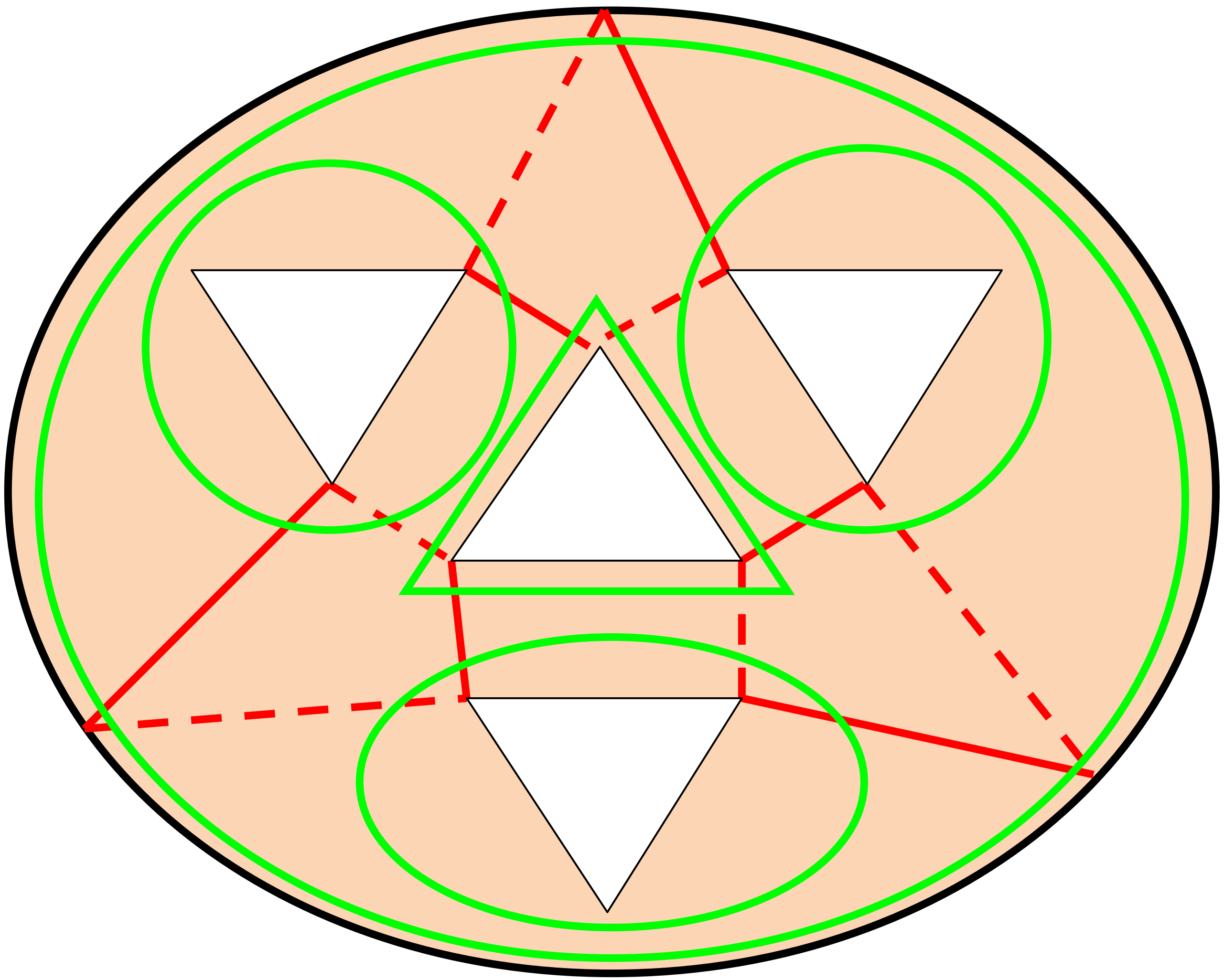
- Consider a plane projection of a knot K in S^3.
- Construct a surface S by thickening this projection.
- Construct a union of simple closed curves of two different colors, red and green, using the following procedure:

The local construction
of a Heegaard diagram
from a knot projection

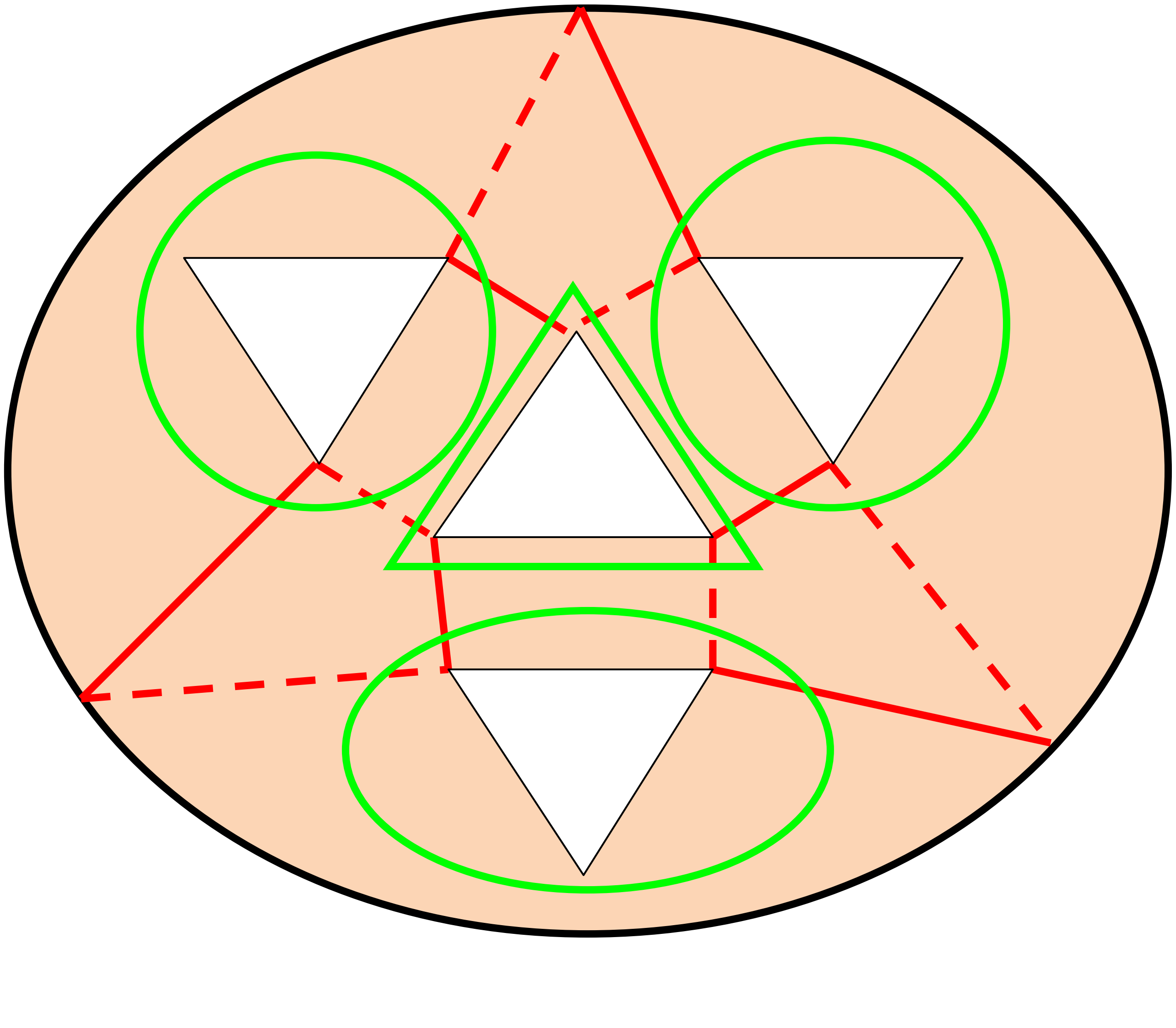


The local construction
of a Heegaard diagram
from a knot projection

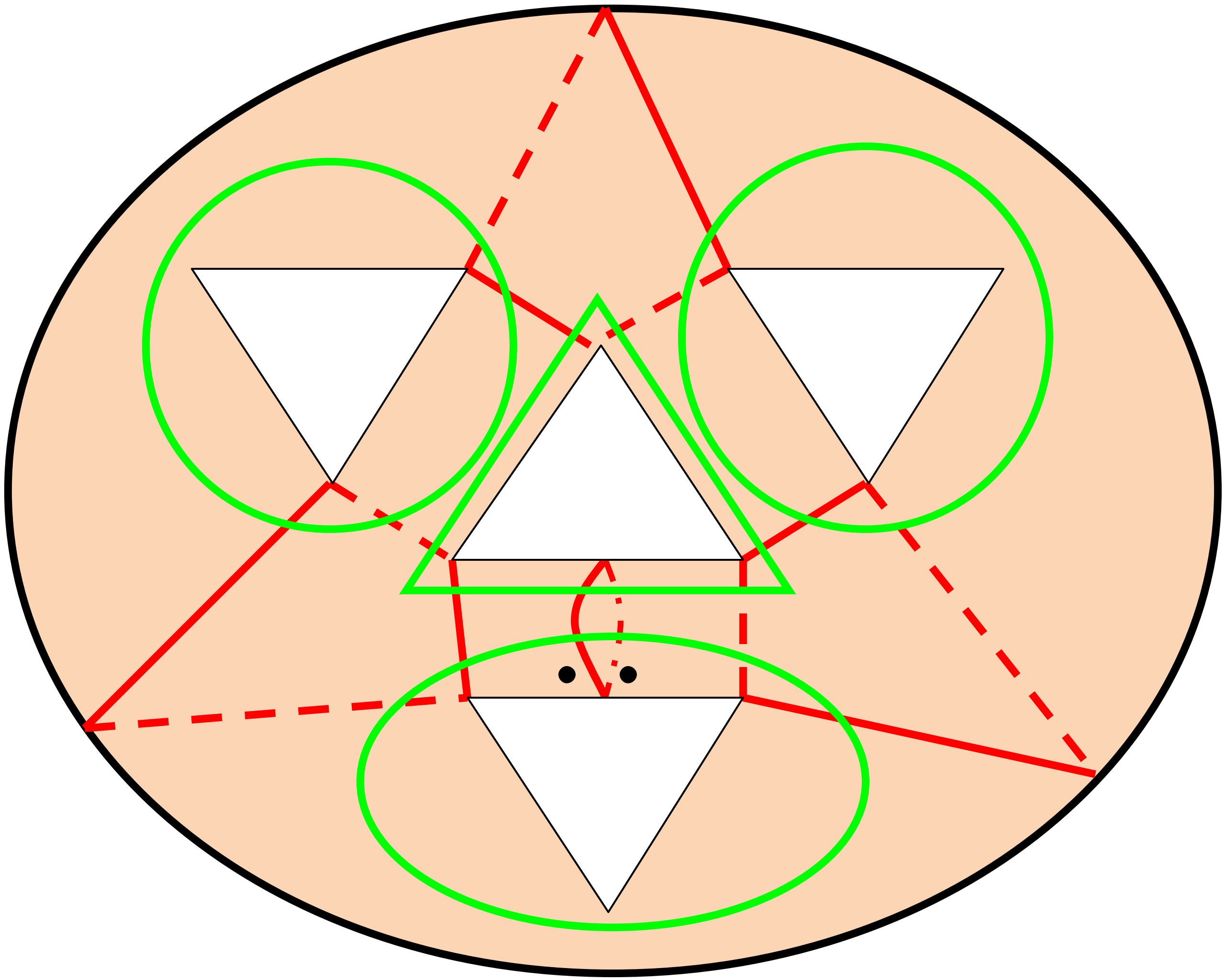




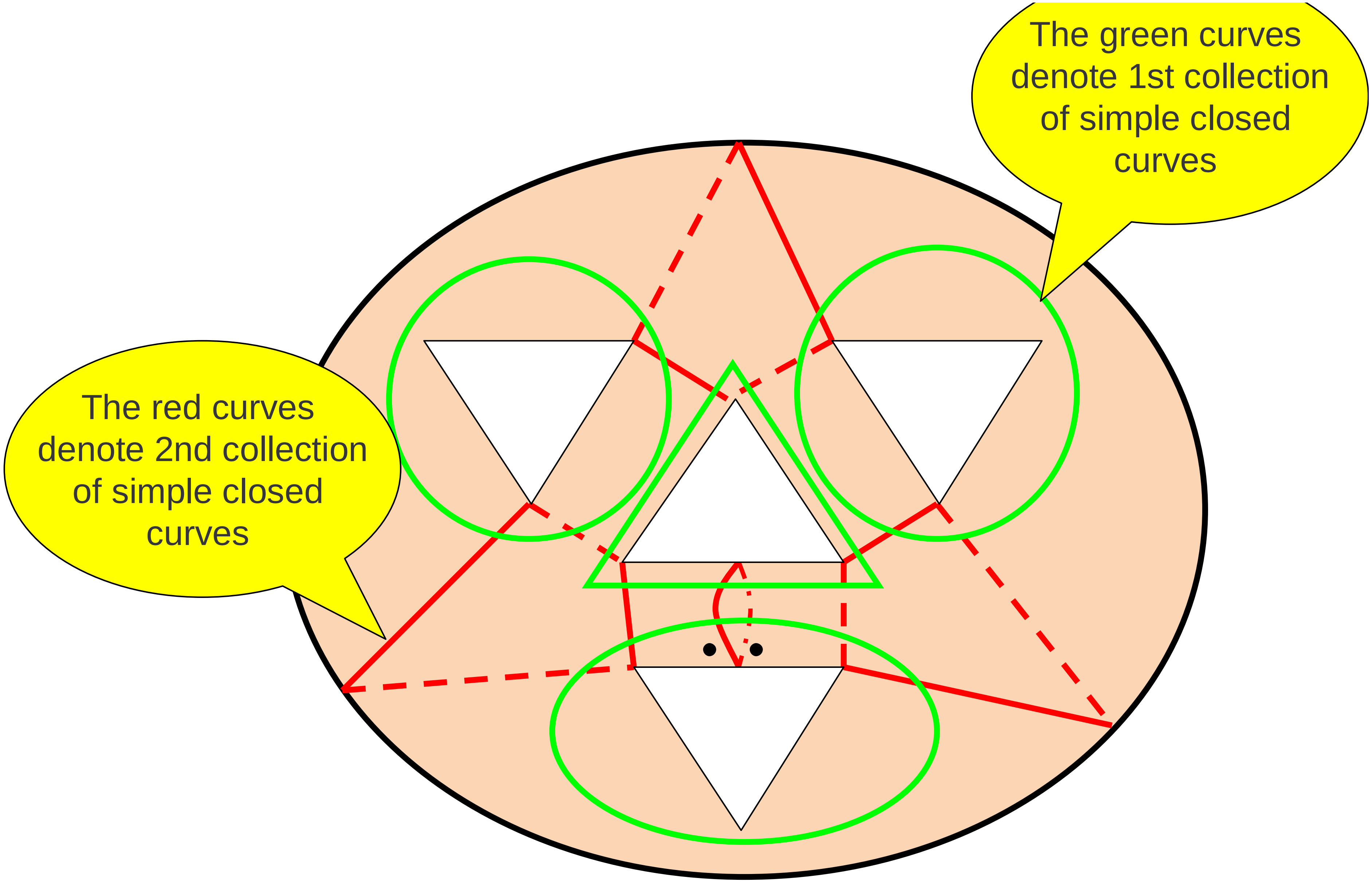
The Heegaard diagram for trefoil after 2nd step



Delete the outer green curve



Add a new red curve and a pair of marked points on its two sides so that the red curve corresponds to the meridian of K.


The green curves denote 1st collection of simple closed curves
The red curves denote 2nd collection of simple closed curves

From topology to Heegaard diagrams

- Using this process we successfully extract a topological structure (a three-manifold, or a knot inside a three-manifold) from a set of combinatorial data: a marked Heegaard diagram

$H=(S, (\alpha_1,\alpha_2,\ldots,\alpha_g),(\beta_1,\beta_2,\ldots,\beta_g),z_1,\ldots,z_n)$

where n is the number of marked points on S.

From Heegaard diagrams to Floer homology

- Heegaard Floer homology associates a homology theory to any Heegaard diagram with marked points.

From Heegaard diagrams to Floer homology

- Heegaard Floer homology associates a homology theory to any Heegaard diagram with marked points.
- In order to obtain an invariant of the topological structure, we should show that if two Heegaard diagrams describe the same topological structure (i.e. 3-manifold or knot), the associated homology groups are isomorphic.

From Heegaard diagrams to Floer homology

- Given a marked Heegaard diagram

  $H=(S, (\alpha_1,\alpha_2,\ldots,\alpha_g),(\beta_1,\beta_2,\ldots,\beta_g),z_1,\ldots,z_n)$, and a ring A which has the structure of a $\mathbf{Z}[u_1,u_2,\ldots,u_n]$-module, Heegaard Floer homology associates a homology group HF(H;A).

From Heegaard diagrams to Floer homology

- Given a marked Heegaard diagram

  $H=(S, (\alpha_1,\alpha_2,\ldots,\alpha_g),(\beta_1,\beta_2,\ldots,\beta_g),z_1,\ldots,z_n)$,
  and a ring A which has the structure of a $\mathbf{Z}[u_1,u_2,\ldots,u_n]$-module, Heegaard Floer homology associates a homology group HF(H;A).
- HF(H;A) is an A-module and is equipped with a **Z**-grading if n=2.

Some results for knots in S^3

- For each $\mathbf{s} \in \mathbf{Z}$, we obtain a group HF(K,**s**) which is an invariant of K (here A=Z).

Some results for knots in S^3

- For each $\mathbf{s} \in \mathbf{Z}$, we obtain a group HF(K,**s**) which is an invariant of K (here A=Z).
- There is a homological grading induced on HF(K,**s**). As a result

$$HF(K,\mathbf{s}) = \bigoplus_{\mathbf{i} \in \mathbf{Z}} HF_{\mathbf{i}}(K,\mathbf{s})$$

Some results for knots in S^3

- For each $\mathbf{s} \in \mathbf{Z}$, we obtain a group HF(K,**s**) which is an invariant of K (here A=Z).
- There is a homological grading induced on HF(K,**s**). As a result

$$HF(K,\mathbf{s}) = \bigoplus_{i \in \mathbf{Z}} HF_i(K,\mathbf{s})$$

- So each HF(K,**s**) has a well-defined Euler characteristic $\chi(K,\mathbf{s})$

Some results for knots in S^3

- The polynomial

$$P_K(t)=\sum_{\mathbf{s}\in \mathbf{Z}} \chi(K,\mathbf{s}).t^s$$

will be the symmetrized Alexander polynomial of K.

Some results for knots in S^3

- The polynomial

  $$P_K(t)=\sum_{\mathbf{s}\in \mathbf{Z}} \chi(K,\mathbf{s}).t^{\mathbf{s}}$$

  will be the symmetrized Alexander polynomial of K.
- There is a symmetry as follows:

  $$HF_i(K,\mathbf{s})=HF_{i-2\mathbf{s}}(K,-\mathbf{s})$$

Some results for knots in S^3

- The polynomial

$$P_K(t)=\sum_{\mathbf{s}\in \mathbf{Z}} \chi(K,\mathbf{s}).t^{s}$$

will be the symmetrized Alexander polynomial of K.

- There is a symmetry as follows:

$$HF_i(K,\mathbf{s})=HF_{i-2\mathbf{s}}(K,-\mathbf{s})$$

- HF(K) determines the genus of K as follows;

Genus of a knot

- Suppose that K is a knot in S^3.

Genus of a knot

- Suppose that K is a knot in S^3.
- Consider all the oriented surfaces C with one boundary component in $S^3\backslash K$ such that the boundary of C is K.

Genus of a knot

- Suppose that K is a knot in S^3.
- Consider all the oriented surfaces C with one boundary component in $S^3 \setminus K$ such that the boundary of C is K.
- Such a surface is called a Seifert surface for K.

Genus of a knot

- Suppose that K is a knot in S^3.
- Consider all the oriented surfaces C with one boundary component in $S^3\backslash K$ such that the boundary of C is K.
- Such a surface is called a Seifert surface for K.
- The genus g(K) of K is the minimum genus for a Seifert surface for K.

HFH determines the genus

- Let d(K) be the largest integer **s** such that HF(K,**s**) is non-trivial.

HFH determines the genus

- Let d(K) be the largest integer **s** such that HF(K,**s**) is non-trivial.
- Theorem (Ozsváth-Szabó) For any knot K in S^3, d(K)=g(K).

HFH and the 4-ball genus

- In fact there is a slightly more interesting invariant $\tau(K)$ defined from HF(K,A), where $A=\mathbf{Z}[u_1^{-1},u_2^{-1}]$, which gives a lower bound for the 4-ball genus $g_4(K)$ of K.

HFH and the 4-ball genus

- In fact there is a slightly more interesting invariant $\tau(K)$ defined from HF(K,A), where $A=\mathbf{Z}[u_1^{-1},u_2^{-1}]$, which gives a lower bound for the 4-ball genus $g_4(K)$ of K.
- The 4-ball genus in the smallest genus of a surface in the 4-ball with boundary K in S^3, which is the boundary of the 4-ball.

HFH and the 4-ball genus

- The 4-ball genus gives a lower bound for the un-knotting number u(K) of K.

HFH and the 4-ball genus

- The 4-ball genus gives a lower bound for the un-knotting number u(K) of K.
- Theorem(Ozsváth-Szabó)

$$\tau(K) \leq g_4(K) \leq u(K)$$

HFH and the 4-ball genus

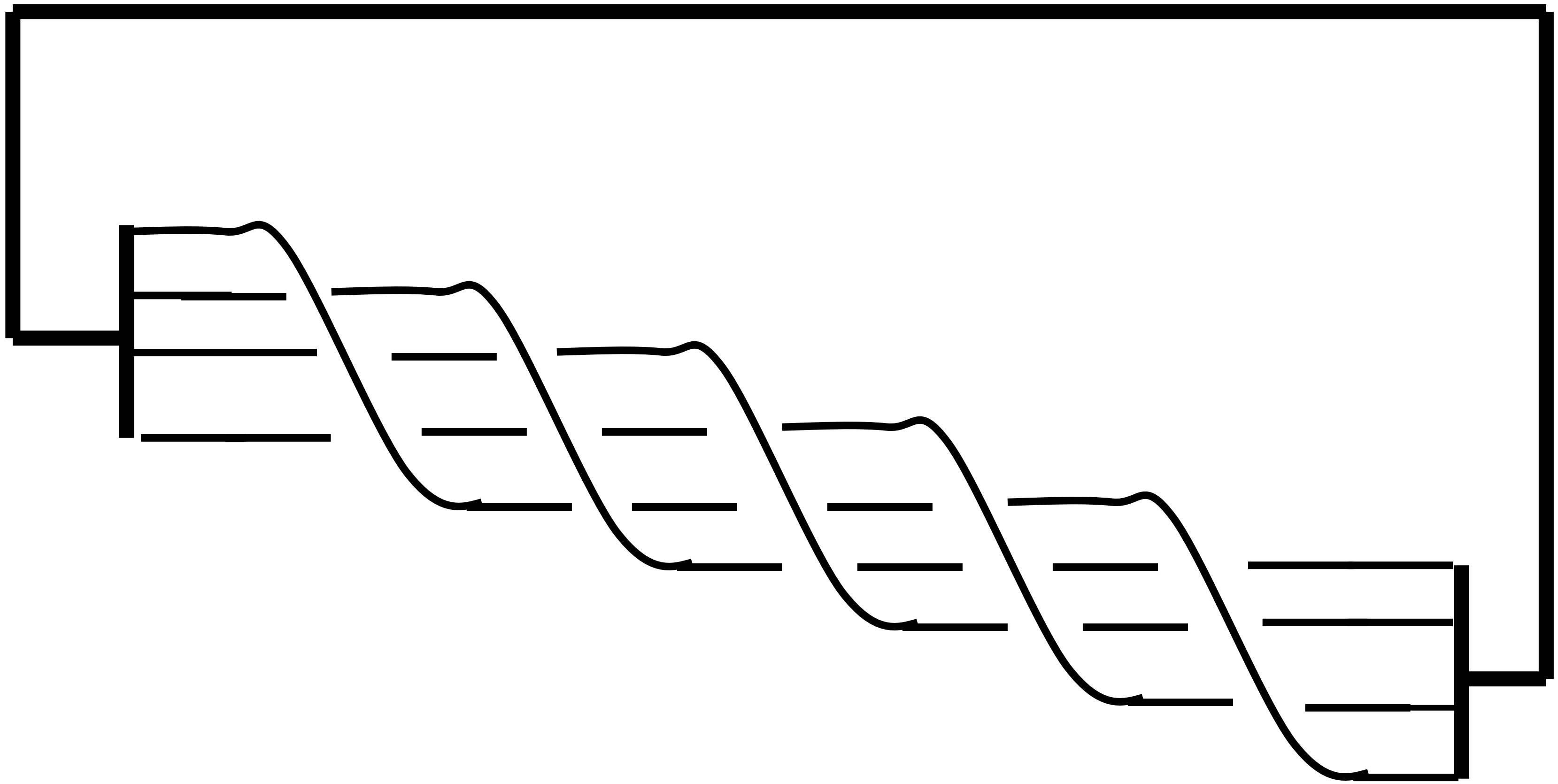
- The 4-ball genus gives a lower bound for the un-knotting number u(K) of K.
- Theorem(Ozsváth-Szabó)

$$\tau(K) \leq g_4(K) \leq u(K)$$

- Corollary(Milnor conjecture, 1st proved by Kronheimer-Mrowka using gauge theory)

If T(p,q) denotes the (p,q) torus knot, then u(T(p,q))=(p-1)(q-1)/2

T(p,q): p strands, q twists



3-manioflds with trivial HF

- For Y=S^3, HF(Y,A)=A.

3-manioflds with trivial HF

- For $Y=S^3$, HF(Y,A)=A.
- For A=**Z** there is a combinatorial algorithm for computing HF(Y,**Z**) for any given 3-manifold from its Heegaard diagram.

3-manioflds with trivial HF

- For $Y=S^3$, HF(Y,A)=A.
- For A=**Z** there is a combinatorial algorithm for computing HF(Y,**Z**) for any given 3-manifold from its Heegaard diagram.
- Question: *Are there other 3-manifolds with trivial Heegaard Floer homology?*

3-manioflds with trivial HF

- The structures on HFH imply that if HF(Y;A) is trivial, then the ordinary homology of Y is the same as the ordinary homology of S^3.

3-manioflds with trivial HF

- The structures on HFH imply that if HF(Y;A) is trivial, then the ordinary homology of Y is the same as the ordinary homology of S^3.
- Such 3-manifolds are called homology spheres.

3-manioflds with trivial HF

- The structures on HFH imply that if HF(Y;A) is trivial, then the ordinary homology of Y is the same as the ordinary homology of S^3.
- Such 3-manifolds are called homology spheres.
- If X=Y#Z and HF(X) is trivial, then both HF(Y) and HF(Z) are trivial.

3-manioflds with trivial HF

- The structures on HFH imply that if HF(Y;A) is trivial, then the ordinary homology of Y is the same as the ordinary homology of S^3.
- Such 3-manifolds are called homology spheres.
- If X=Y#Z and HF(X) is trivial, then both HF(Y) and HF(Z) are trivial.
- Theorem(E.) If for a prime homology sphere Y we have HF(Y;Z)=Z, then Y can not contain an incompressible torus.

3-manioflds with trivial HF

- The structures on HFH imply that if HF(Y;A) is trivial, then the ordinary homology of Y is the same as the ordinary homology of S^3.
- Such 3-manifolds are called homology spheres.
- If X=Y#Z and HF(X) is trivial, then both HF(Y) and HF(Z) are trivial.
- Theorem(E.) If for a prime homology sphere Y we have HF(Y;Z)=Z, then Y can not contain an incompressible torus.

3-maniouflds with trivial HF

- Thurston Geometrization (Perelman): If Y is a prime 3-manifold without any incompressible torus inside it, then Y is either hyperbolic, or has one of the other 7 geometries of Thurston.

3-manioflds with trivial HF

- Thurston Geometrization (Perelman): If Y is a prime 3-manifold without any incompressible torus inside it, then Y is either hyperbolic, or has one of the other 7 geometries of Thurston.
- Theorem (E.). If Y is a homology sphere which has one of the 7 other geometries of Thurston and HF(Y,**Z**)=**Z**, then Y is either S^3 or the Poincare sphere P. Moreover, HF(P,A)=A for all A!

3-manioflds with trivial HF

- Conjecture. If Y is a hyperbolic homology sphere, then HF(Y,**Z**) is not equal trivial (i.e. equal to **Z**).

3-manioflds with trivial HF

- Conjecture. If Y is a hyperbolic homology sphere, then HF(Y,**Z**) is not equal trivial (i.e. equal to **Z**).
- If the conjecture is true, the only 3-manifolds with trivial Heegaard Floer homology are proved to be connected sums of several copies of the Poincare sphere.

Main construction of HFH

- Fix a Heegaard diagram

  $H=(S, (\alpha_1,\alpha_2,\ldots,\alpha_g),(\beta_1,\beta_2,\ldots,\beta_g),z_1,\ldots,z_n)$

Main construction of HFH

- Fix a Heegaard diagram

  $H=(S, (\alpha_1,\alpha_2,\ldots,\alpha_g),(\beta_1,\beta_2,\ldots,\beta_g),z_1,\ldots,z_n)$
- Construct the complex 2g-dimensional smooth manifold

  $X=Sym^g(S)=(S\times S\times\ldots\times S)/S(g)$

  where S(g) is the permutation group on g letters acting on the g-tuples of points from S.

Main construction of HFH

- Fix a Heegaard diagram

  $H=(S, (\alpha_1,\alpha_2,\ldots,\alpha_g),(\beta_1,\beta_2,\ldots,\beta_g),z_1,\ldots,z_n)$
- Construct the complex 2g-dimensional smooth manifold

  $X=Sym^g(S)=(S\times S\times\ldots\times S)/S(g)$

  where S(g) is the permutation group on g letters acting on the g-tuples of points from S.
- Every complex structure on S determines a complex structure on X.

Main construction of HFH

- Consider the two g-dimensional tori

$T_\alpha=\alpha_1\times\alpha_2\times\ldots\times\alpha_g$ and $T_\beta=\beta_1\times\beta_2\times\ldots\times\beta_g$ in $Z=S\times S\times\ldots\times S$. The projection map from Z to X embeds these two tori in X.

Main construction of HFH

- Consider the two g-dimensional tori

$T_\alpha=\alpha_1\times\alpha_2\times\ldots\times\alpha_g$ and $T_\beta=\beta_1\times\beta_2\times\ldots\times\beta_g$

in $Z=S\times S\times\ldots\times S$. The projection map from Z to X embeds these two tori in X.

- These tori are totally real sub-manifolds of the complex manifold X.

Main construction of HFH

- Consider the two g-dimensional tori

  $T_\alpha=\alpha_1\times\alpha_2\times\ldots\times\alpha_g$ and $T_\beta=\beta_1\times\beta_2\times\ldots\times\beta_g$

  in $Z=S\times S\times\ldots\times S$. The projection map from Z to X embeds these two tori in X.
- These tori are totally real sub-manifolds of the complex manifold X.
- If the curves $\alpha_1,\alpha_2,\ldots,\alpha_g$ meet the curves $\beta_1,\beta_2,\ldots,\beta_g$ transversally on S, T_α will meet T_β transversally in X.

Intersection points of T_α and T_β

- A point of intersection between T_α and T_β consists of a g-tuple of points $(x_1,x_2,\ldots,x_g)$ such that for some element $\sigma \in S(g)$ we have $x_i \in \alpha_i \cap \beta_{\sigma(i)}$ for $i=1,2,\ldots,g$.

Intersection points of T_α and T_β

- A point of intersection between T_α and T_β consists of a g-tuple of points $(x_1,x_2,\ldots,x_g)$ such that for some element $\sigma \in S(g)$ we have $x_i \in \alpha_i \cap \beta_{\sigma(i)}$ for i=1,2,…,g.
- The complex CF(H), associated with the Heegaard diagram H, is generated by the intersection points $\mathbf{x}= (x_1,x_2,\ldots,x_g)$ as above.

The coefficient ring will be denoted by A, which is a $\mathbf{Z}[u_1,u_2,\ldots,u_n]$-module.

Differential of the complex

- The differential of this complex should have the following form:

$$d(\mathbf{x}) = \sum_{\mathbf{y} \in T_\alpha \cap T_\beta} b(\mathbf{x},\mathbf{y}).\mathbf{y}$$

The values b(**x,y**)$\in$A should be determined. Then d may be linearly extended to CF(H).

Differential of the complex; b(**x**,**y**)

- For **x**,**y**$\in T_\alpha \cap T_\beta$ consider the space π_2(**x**,**y**) of the homotopy types of the disks satisfying the following properties:

$$u:[0,1]\times\mathbf{R}\subset\mathbf{C}\to X$$

$$u(0,t)\in T_\alpha,\ u(1,t)\in T_\beta$$

$$u(s,\infty)=\mathbf{x},\ u(s,-\infty)=\mathbf{y}$$

Differential of the complex; b(**x**,**y**)

- For **x**,**y**$\in T_\alpha \cap T_\beta$ consider the space π_2(**x**,**y**) of the homotopy types of the disks satisfying the following properties:

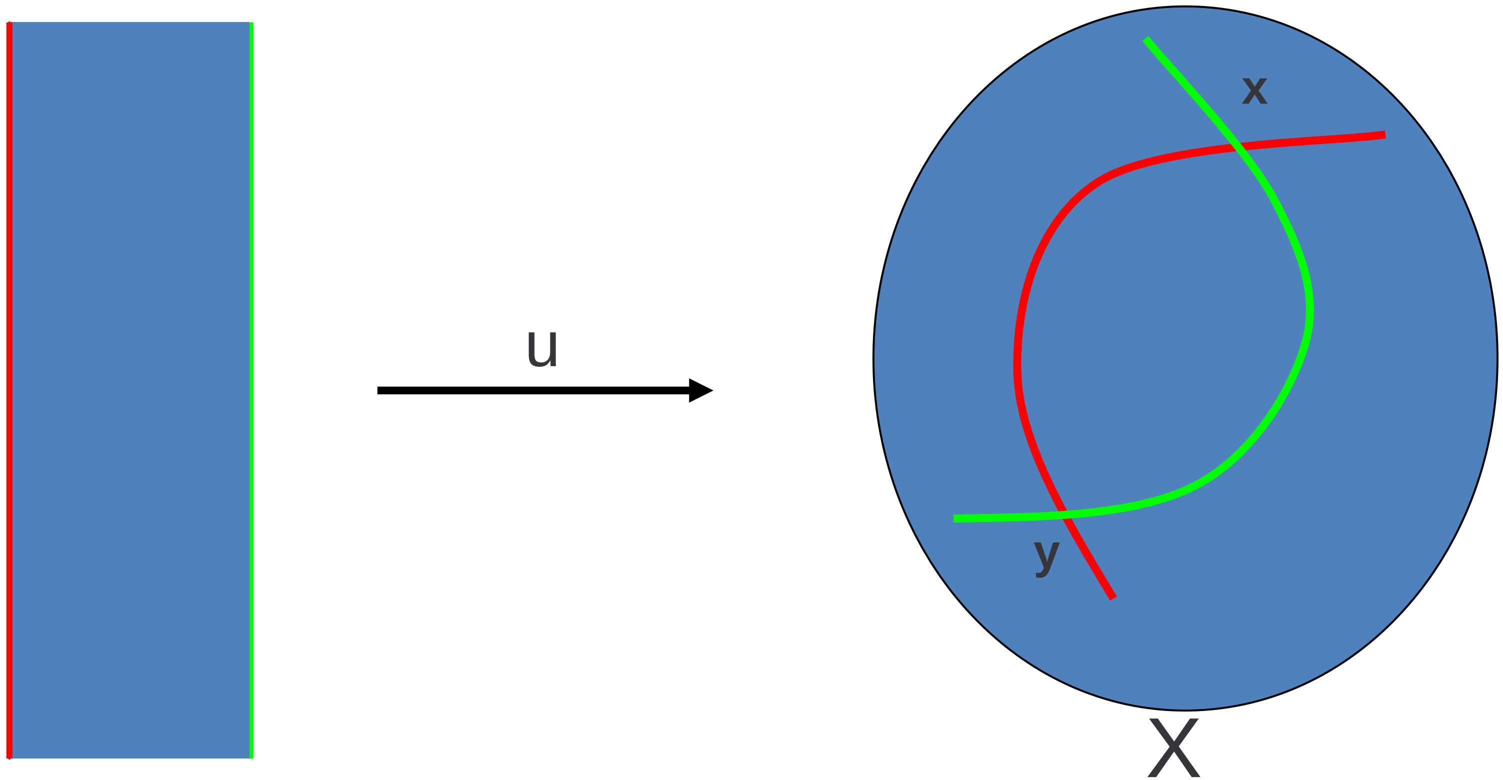
$$u:[0,1]\times \mathbf{R} \subset \mathbf{C} \to X$$

$$u(0,t)\in T_\alpha,\ u(1,t)\in T_\beta$$

$$u(s,\infty)=\mathbf{x},\ u(s,-\infty)=\mathbf{y}$$

- For each $\phi\in\pi_2$(**x**,**y**) let $M(\phi)$ denote the moduli space of holomorphic maps u as above representing the class ϕ.

Differential of the complex; b(**x**,**y**)



Differential of the complex; b(**x**,**y**)

- There is an action of **R** on the moduli space M(ϕ) by translation of the second component by a constant factor: If u(s,t) is holomorphic, then u(s,t+c) is also holomorphic.

Differential of the complex; b(**x**,**y**)

- There is an action of **R** on the moduli space M(ϕ) by translation of the second component by a constant factor: If u(s,t) is holomorphic, then u(s,t+c) is also holomorphic.
- If μ(ϕ) denotes the formal dimension or expected dimension of M(ϕ), then the quotient moduli space is expected to be of dimension μ(ϕ)-1. We may manage to achieve the correct dimension.

Differential of the complex; b(**x**,**y**)

- Let n(ϕ) denote the number of points in the quotient moduli space (counted with a sign) if μ(ϕ)=1. Otherwise define n(ϕ)=0.

Differential of the complex; b(**x**,**y**)

- Let $n(\phi)$ denote the number of points in the quotient moduli space (counted with a sign) if $\mu(\phi)=1$. Otherwise define $n(\phi)=0$.
- Let $n(j,\phi)$ denote the intersection number

of $L(z_j)=\{z_j\}\times Sym^{g-1}(S)\subset Sym^g(S)=X$

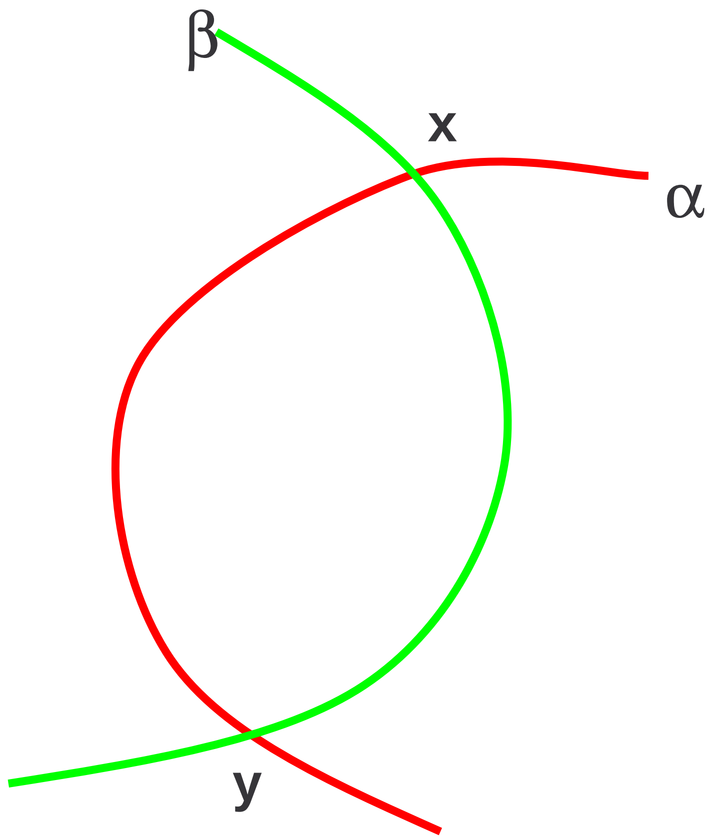
with ϕ.

Differential of the complex; b($\mathbf{x}$,$\mathbf{y}$)

- Let n(ϕ) denote the number of points in the quotient moduli space (counted with a sign) if $\mu(\phi)=1$. Otherwise define $n(\phi)=0$.
- Let n(j,ϕ) denote the intersection number of $L(z_j)=\{z_j\}\times Sym^{g-1}(S)\subset Sym^{g}(S)=X$ with ϕ.
- Define $b(\mathbf{x},\mathbf{y})=\sum_{\phi} n(\phi).\prod_j u_j^{\,n(j,\phi)}$ where the sum is over all $\phi\in\pi_2(\mathbf{x},\mathbf{y})$.

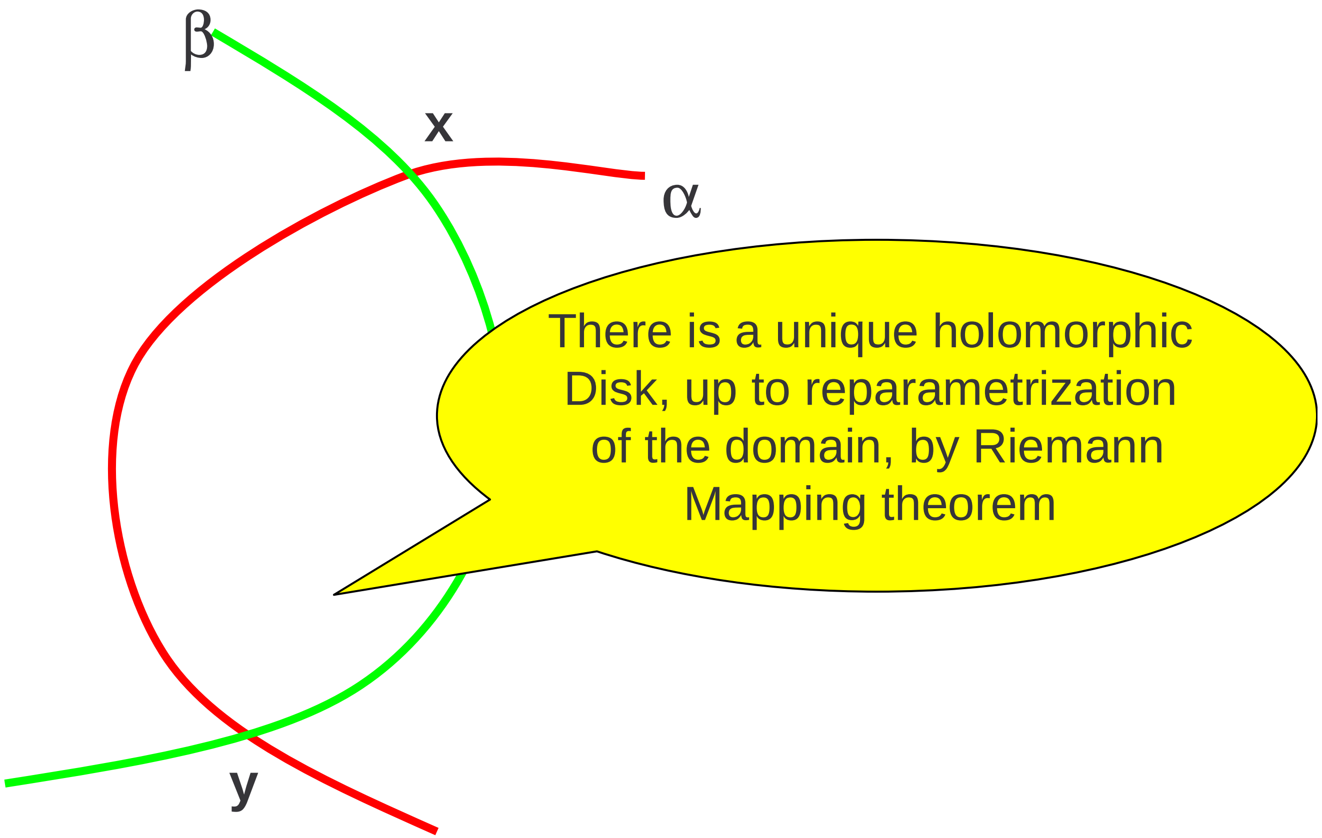
Two examples in dimension two

Example 1.



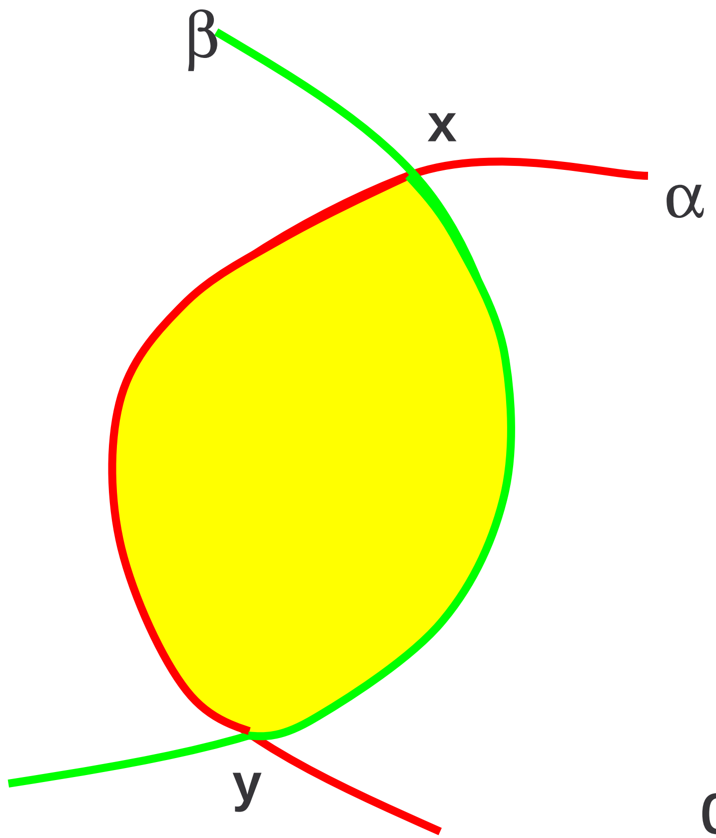
Two examples in dimension two

Example 1.



Two examples in dimension two

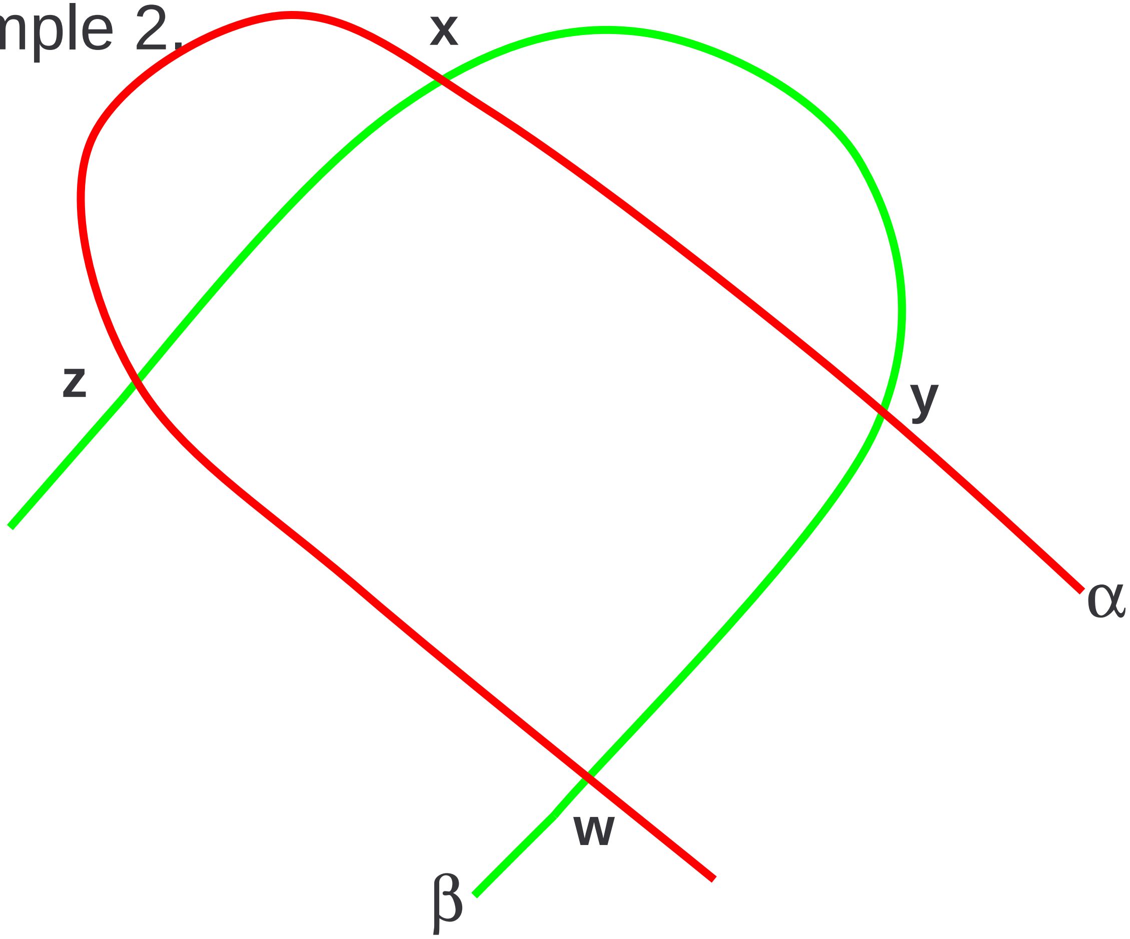
Example 1.



d(x)=y

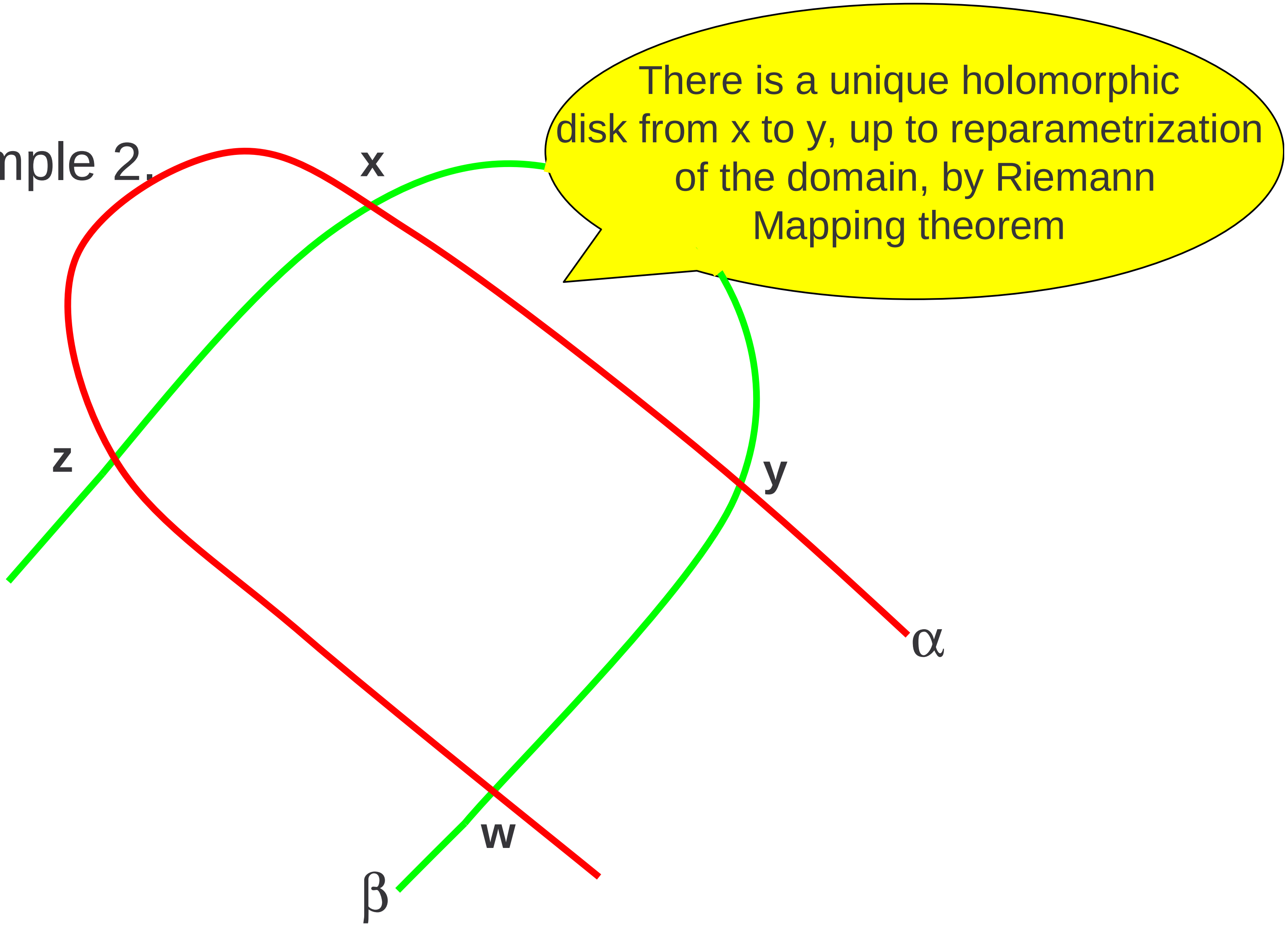
Two examples in dimension two

Example 2.



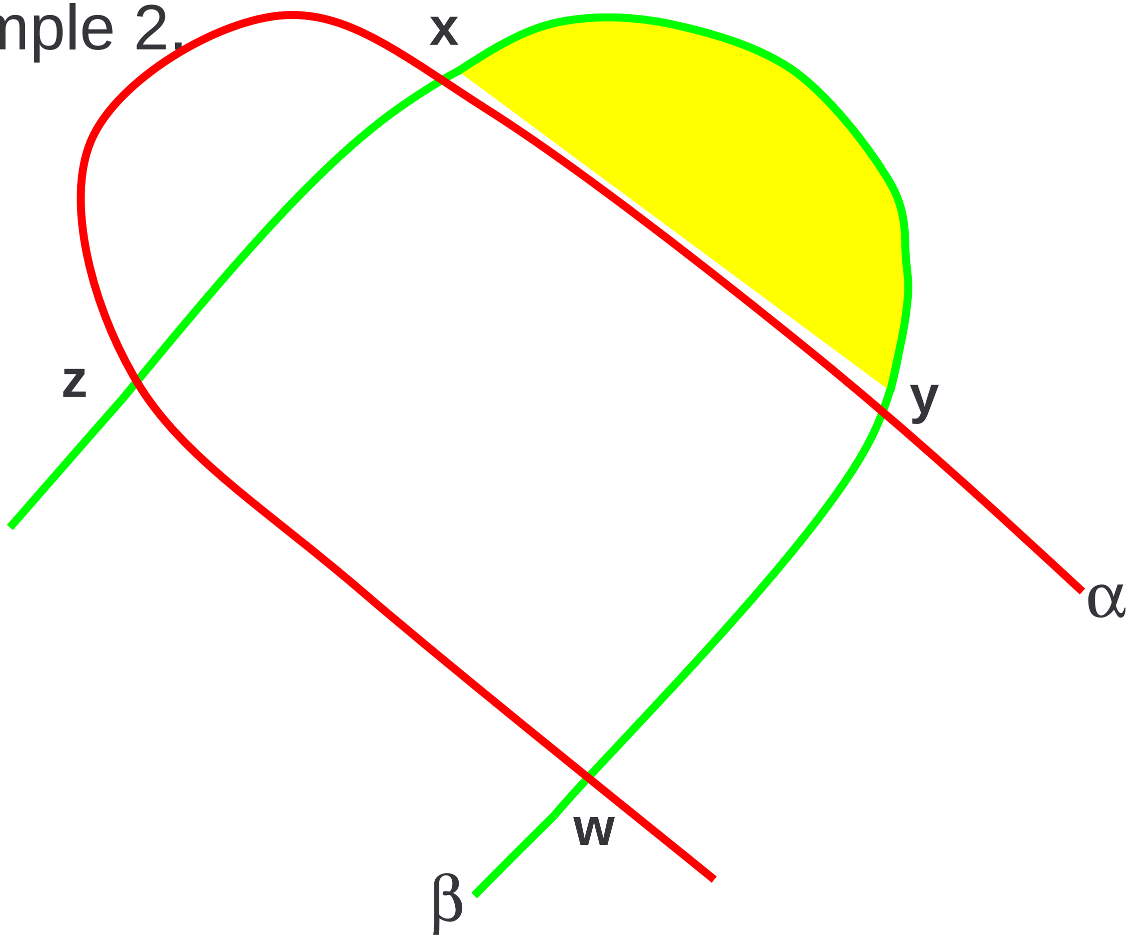
Two examples in dimension two

Example 2.



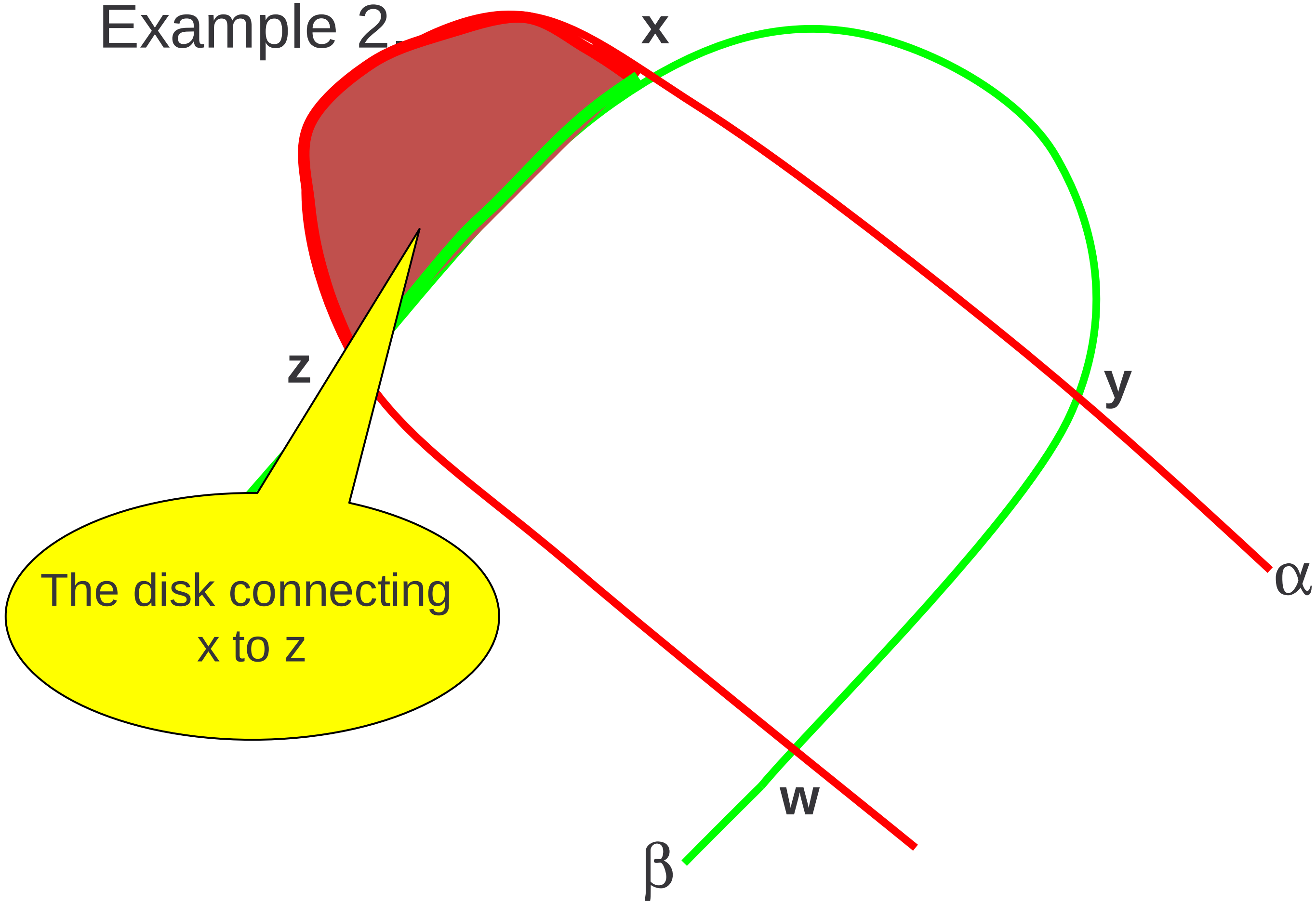
Two examples in dimension two

Example 2.



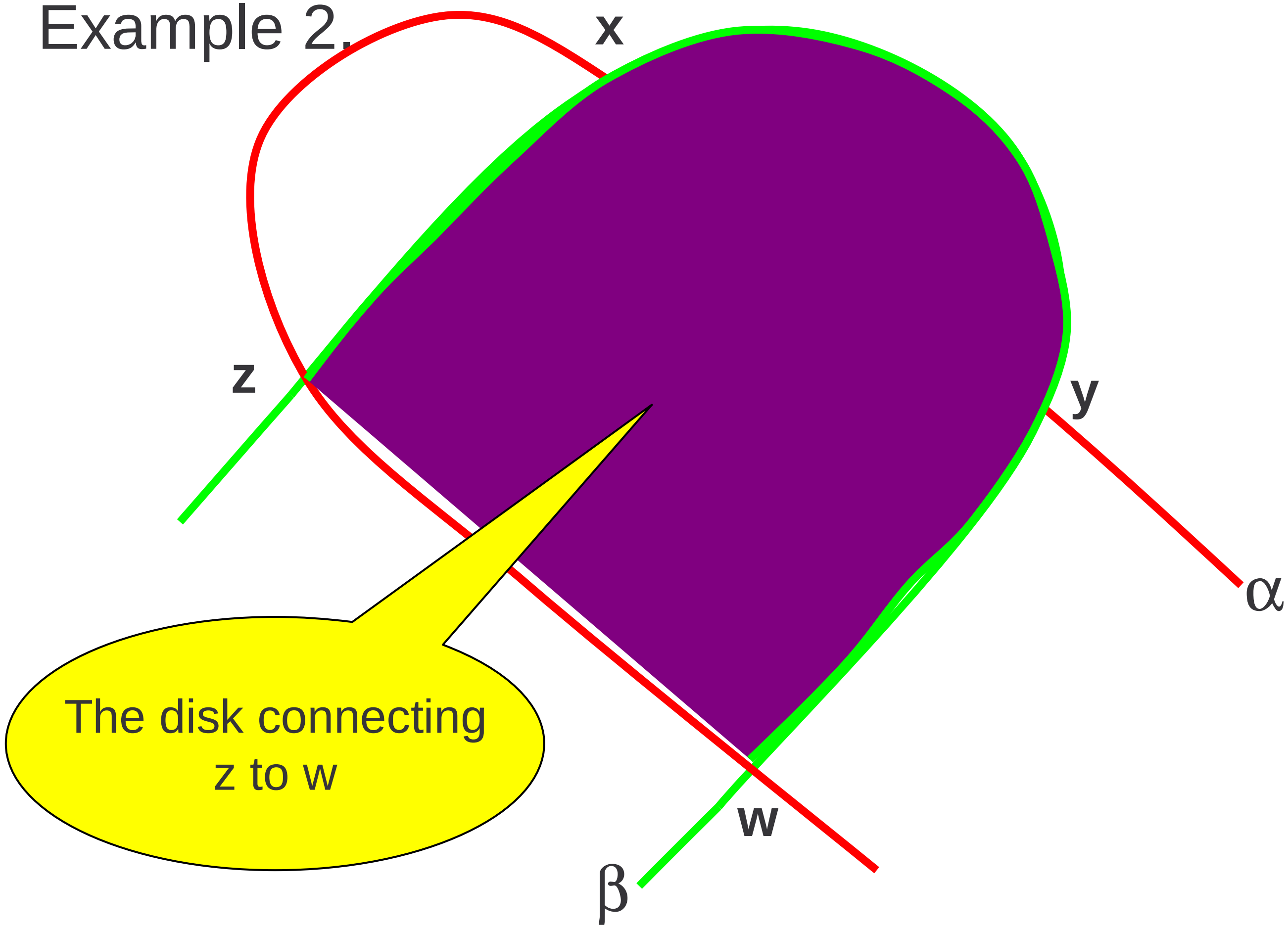
Two examples in dimension two

Example 2.



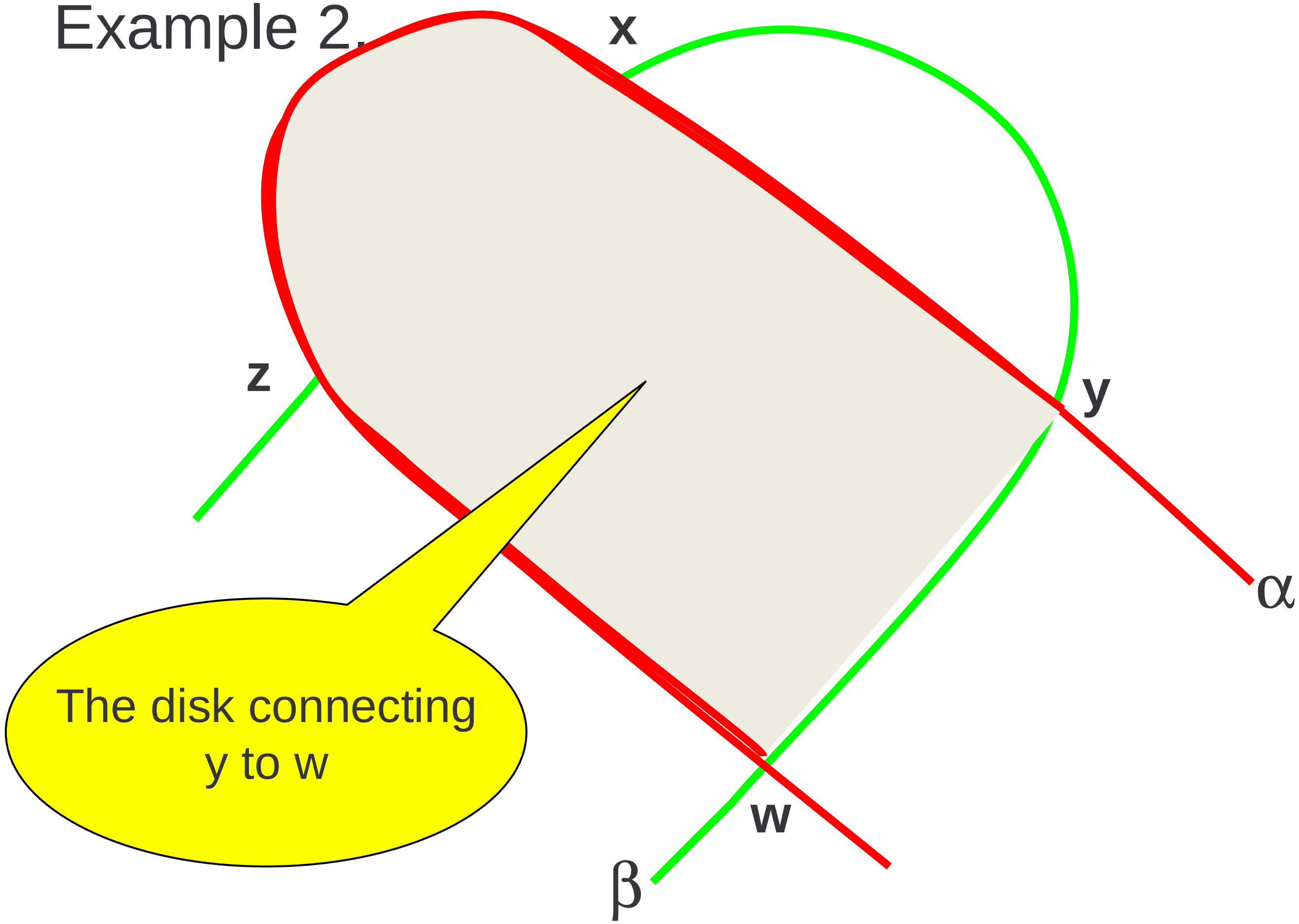
Two examples in dimension two

Example 2.



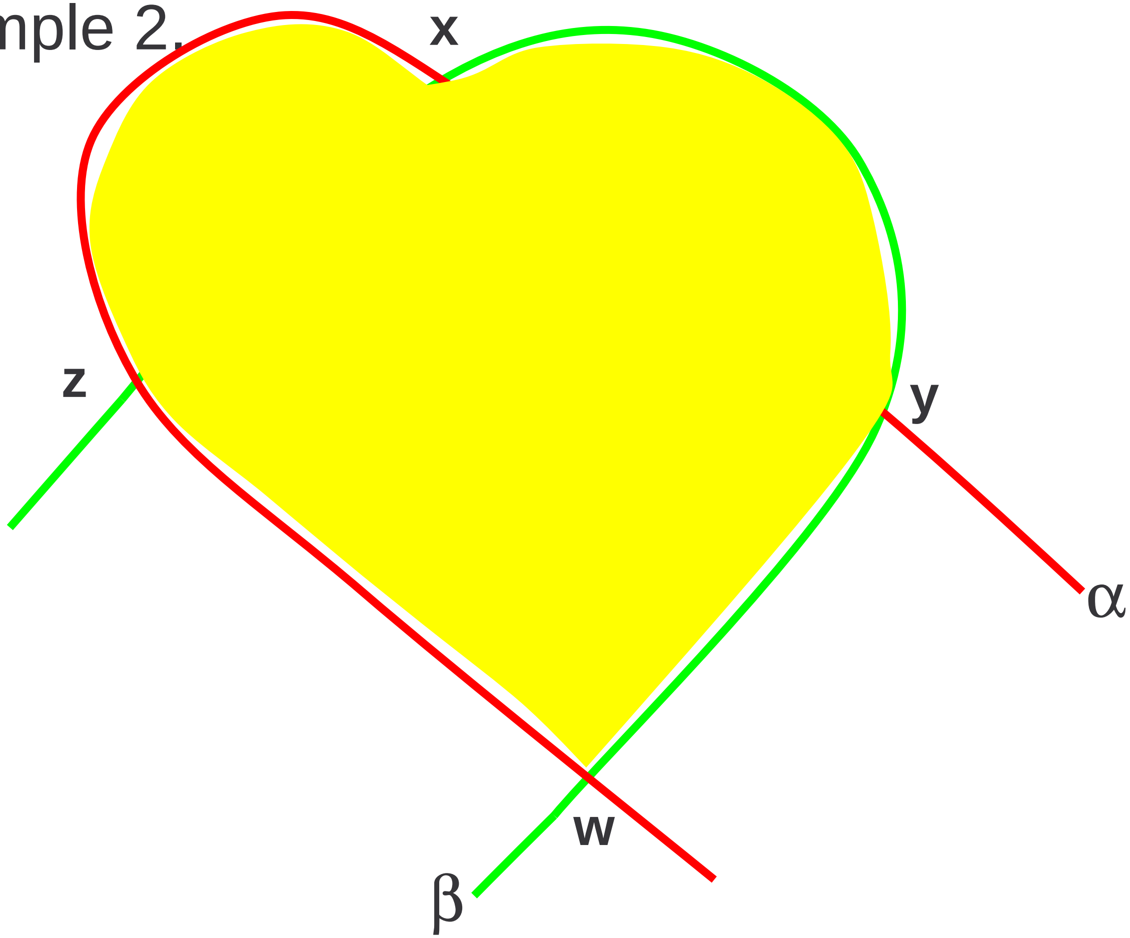
Two examples in dimension two

Example 2.



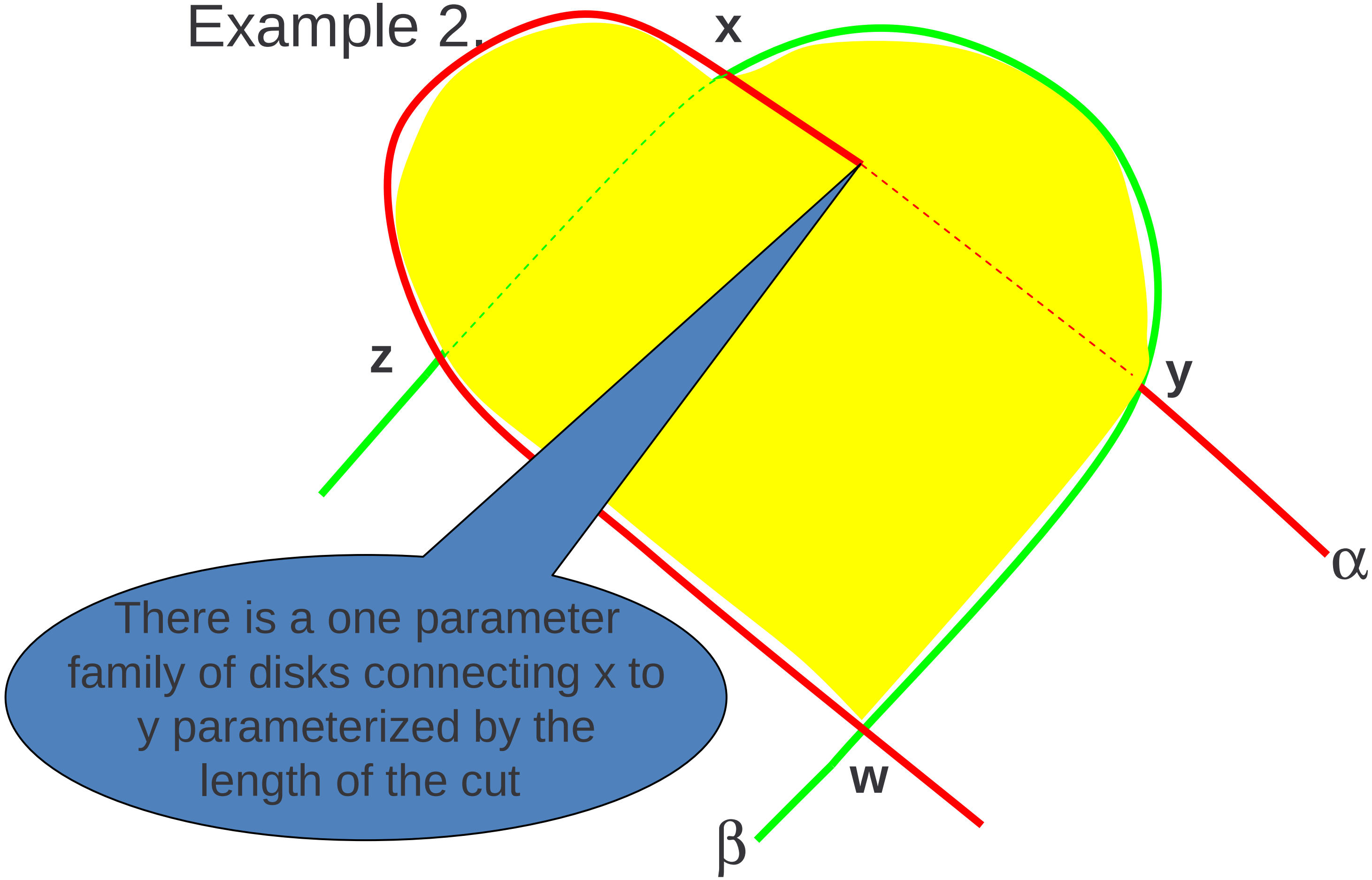
Two examples in dimension two

Example 2.



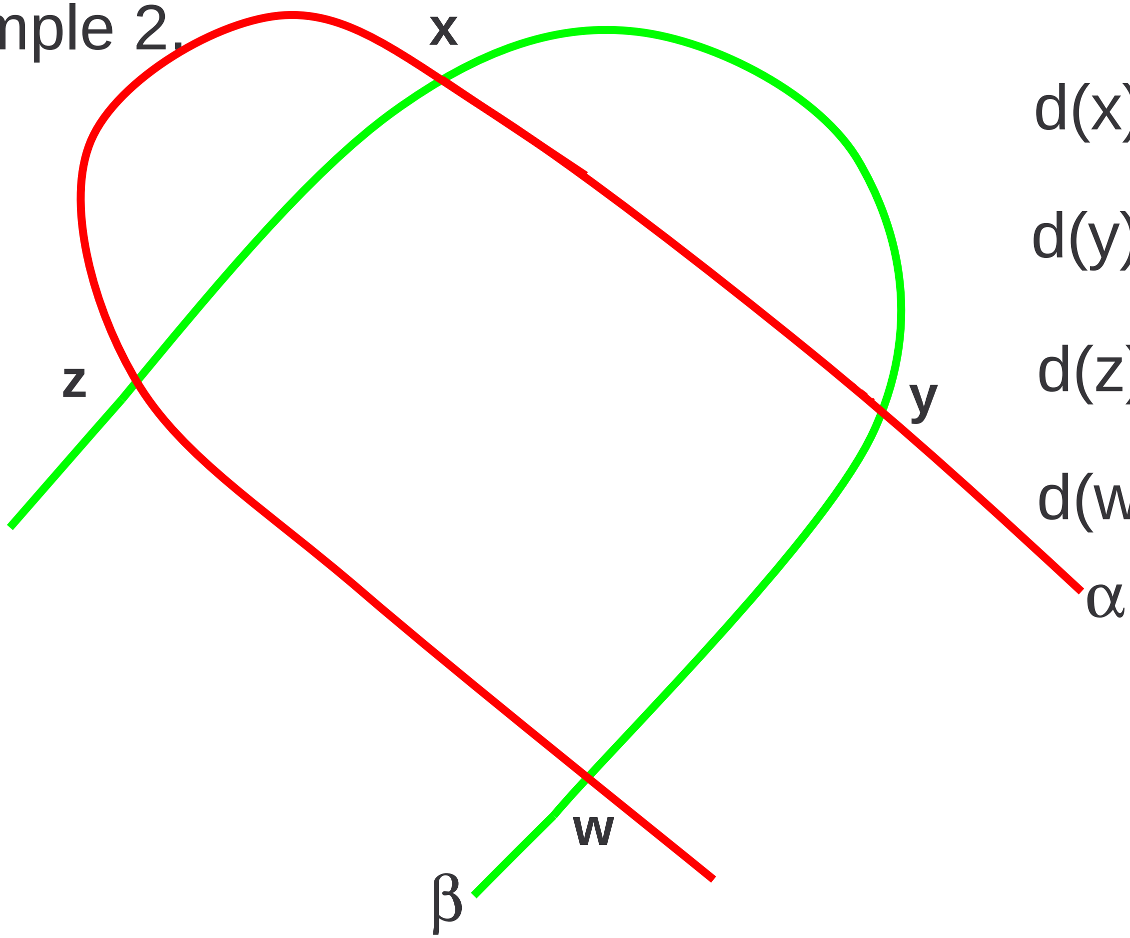
Two examples in dimension two

Example 2.



Two examples in dimension two

Example 2.



d(x)=y+z

d(y)=w

d(z)=-w

d(w)=0

Basic properties

- The first observation is that $d^2=0$.

Basic properties

- The first observation is that $d^2=0$.
- This may be checked easily in the two examples discussed here.

Basic properties

- The first observation is that $d^2=0$.
- This may be checked easily in the two examples discussed here.
- In general the proof uses a description of the boundary of $M(\phi)/\sim$ when $\mu(\phi)=2$. Here $\sim$ denotes the equivalence relation obtained by **R**-translation. Gromov compactness theorem and a gluing lemma should be used.

Basic properties

- <u>Theorem</u> (Ozsváth-Szabó) The homology groups HF(H,A) of the complex (CF(H),d) are invariants of the pointed Heegaard diagram H. For a three-manifold Y, or a knot ($K \subset Y$), the homology group is in fact independent of the specific Heegaard diagram used for constructing the chain complex and gives homology groups HF(Y,A) and HFK(K,A) respectively.

Refinements of these homology groups

- Consider the space $Spin^c(Y)$ of $Spin^c$-structures on Y. This is the space of homology classes of nowhere vanishing vector fields on Y. Two non-vanishing vector fields on Y are called homologous if they are isotopic in the complement of a ball in Y.

Refinements of these homology groups

- Consider the space $\mathrm{Spin}^c(Y)$ of Spin^c-structures on Y. This is the space of homology classes of nowhere vanishing vector fields on Y. Two non-vanishing vector fields on Y are called homologous if they are isotopic in the complement of a ball in Y.
- The marked point z defines a map $\mathbf{s}_z$ from the set of generators of CF(H) to $\mathrm{Spin}^c(Y)$:

$$\mathbf{s}_z : T_\alpha \cap T_\beta \rightarrow \mathrm{Spin}^c(Y)$$

defined as follows :

Refinements of these homology groups

- If $\mathbf{x}=(x_1,x_2,\ldots,x_g)\in T_\alpha \cap T_\beta$ is an intersection point, then each of x_j determines a flow line for the Morse function h connecting one of the index-1 critical points to an index-2 critical point. The marked point z determines a flow line connecting the index-0 critical point to the index-3 critical point.
- All together we obtain a union of flow lines joining pairs of critical points of indices of different parity.

Refinements of these homology groups

- If $\mathbf{x}=(x_1,x_2,\ldots,x_g)\in T_\alpha \cap T_\beta$ is an intersection point, then each of x_j determines a flow line for the Morse function h connecting one of the index-1 critical points to an index-2 critical point. The marked point z determines a flow line connecting the index-0 critical point to the index-3 critical point.
- All together we obtain a union of flow lines joining pairs of critical points of indices of different parity.

Refinements of these homology groups

- The gradient vector field may be modified in a neighborhood of these paths to obtain a nowhere vanishing vector field on Y.

Refinements of these homology groups

- The gradient vector field may be modified in a neighborhood of these paths to obtain a nowhere vanishing vector field on Y.
- The class of this vector field in $\mathrm{Spin}^c(Y)$ is independent of this modification and is denoted by $\mathbf{s}_z(\mathbf{x})$.

Refinements of these homology groups

- The gradient vector field may be modified in a neighborhood of these paths to obtain a nowhere vanishing vector field on Y.
- The class of this vector field in $\mathrm{Spin}^c(Y)$ is independent of this modification and is denoted by $\mathbf{s}_z(\mathbf{x})$.
- If $\mathbf{x},\mathbf{y} \in T_\alpha \cap T_\beta$ are intersection points with $\pi_2(\mathbf{x},\mathbf{y}) \neq \emptyset$, then $\mathbf{s}_z(\mathbf{x}) = \mathbf{s}_z(\mathbf{y})$.

Refinements of these homology groups

- This implies that the homology groups HF(Y,A) decompose according to the $Spin^c$ structures over Y:

$$HF(Y,A)=\bigoplus_{\mathbf{s}\in Spin(Y)} HF(Y,A;\mathbf{s})$$

Refinements of these homology groups

- This implies that the homology groups HF(Y,A) decompose according to the Spin^c structures over Y:

$$HF(Y,A)=\bigoplus_{\mathbf{s}\in \text{Spin}(Y)} HF(Y,A;\mathbf{s})$$

- For each $\mathbf{s}\in \text{Spin}^c(Y)$ the group $HF(Y,A;\mathbf{s})$ is also an invariant of the three-manifold Y and the Spin^c structure **s**.

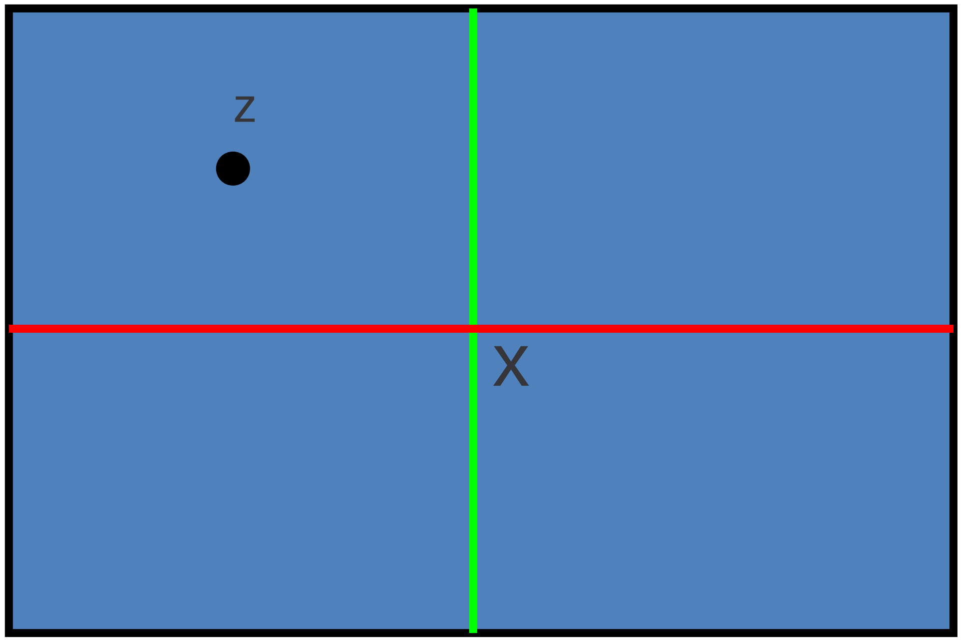
Some examples

- For S^3, $\mathrm{Spin}^c(S^3)=\{\mathbf{s}_0\}$ and $HF(Y,A;\mathbf{s}_0)=A$

Some examples

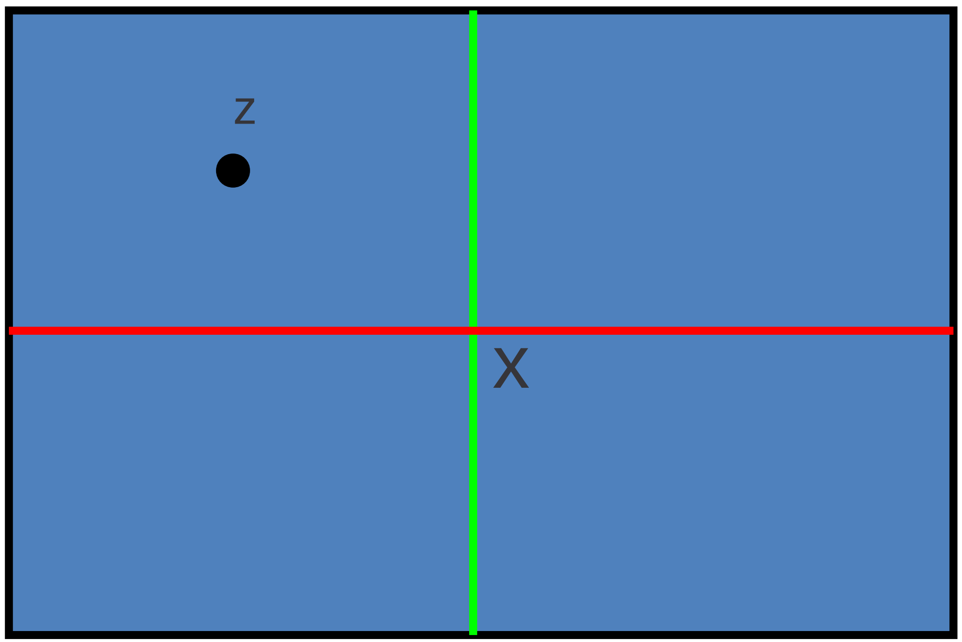
- For S^3, $\mathrm{Spin}^c(S^3)=\{\mathbf{s}_0\}$ and $HF(Y,A;\mathbf{s}_0)=A$
- For $S^1\times S^2$, $\mathrm{Spin}^c(S^1\times S^2)=\mathbf{Z}$. Let $\mathbf{s}_0$ be the Spin^c structure such that $c_1(\mathbf{s}_0)=0$, then for $\mathbf{s}\neq\mathbf{s}_0$, $HF(Y,A;\mathbf{s})=0$. Furthermore we have $HF(Y,A;\mathbf{s}_0)=A\oplus A$, where the homological gradings of the two copies of A differ by 1.

Heegaard diagram for S^3



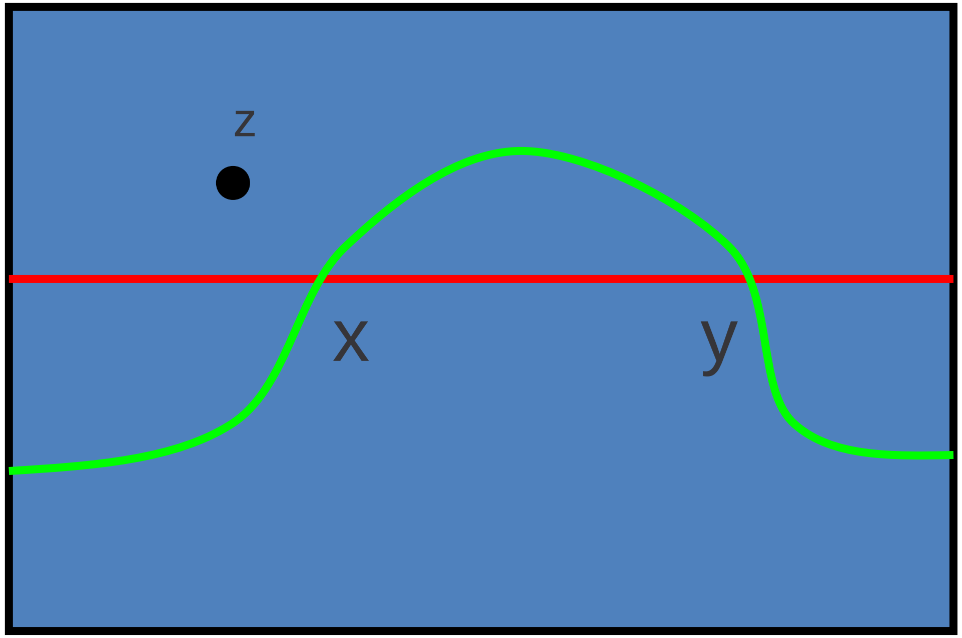
The opposite sides of the rectangle should be identified to obtain a torus (surface of genus1)

Heegaard diagram for S^3



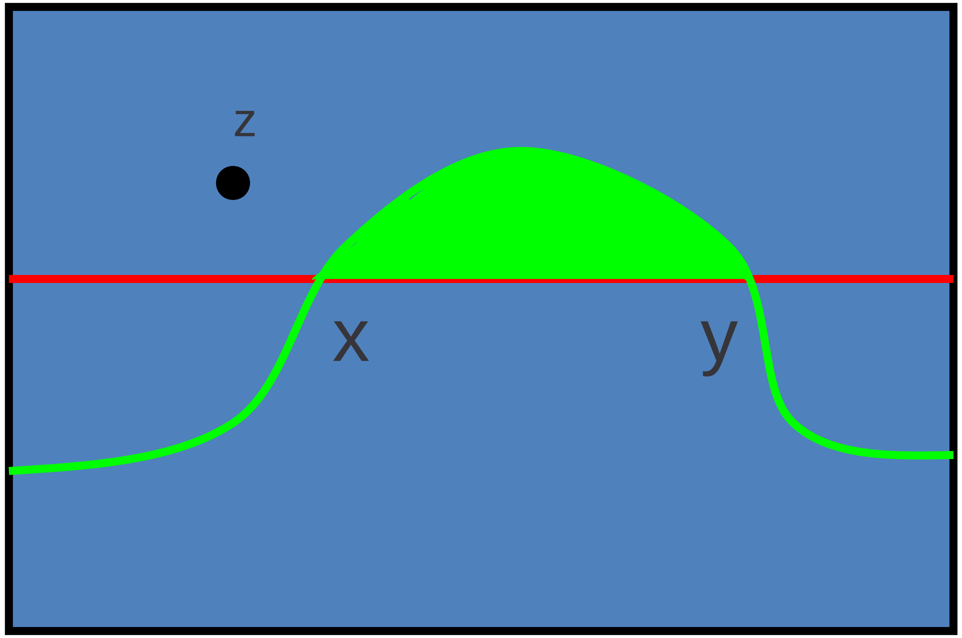
Only one generator x, and no differentials; so the homology will be A

Heegaard diagram for $S^1 \times S^2$



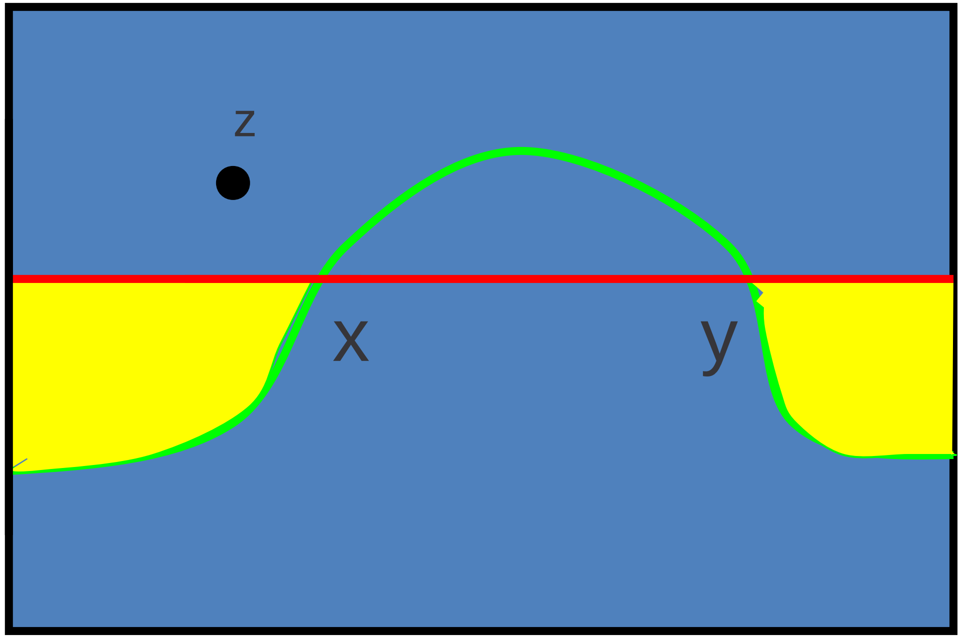
Only two generators x,y and two homotopy classes of disks of index 1.

Heegaard diagram for $S^1 \times S^2$



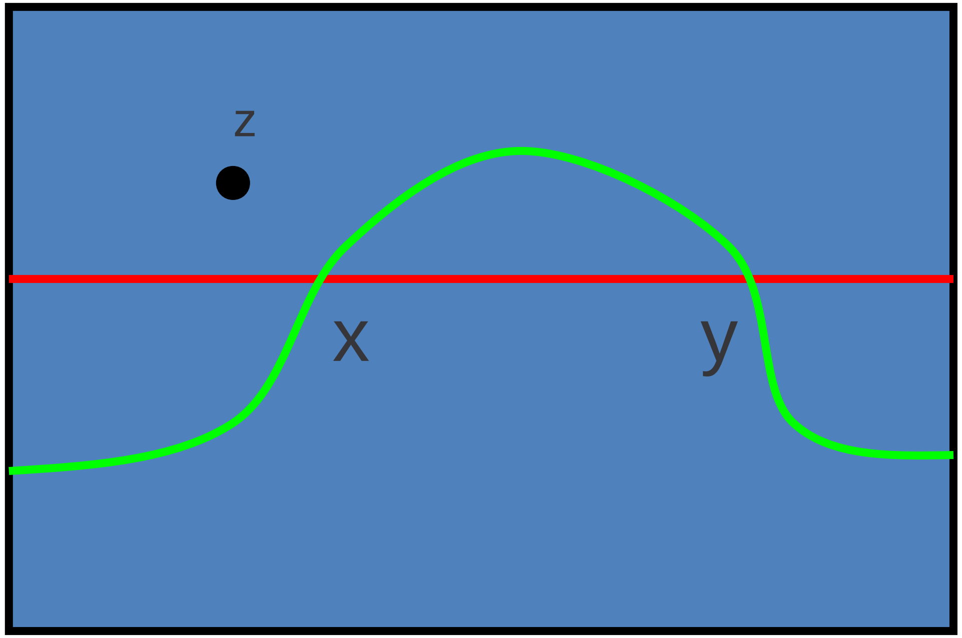
The first disk connecting x to y, with Maslov index one.

Heegaard diagram for $S^1 \times S^2$



The second disk connecting x to y, with Maslov index one. The sign will be different from the first one.

Heegaard diagram for $S^1 \times S^2$



$d(x)=d(y)=0$

$\mathbf{s}_z(x)=\mathbf{s}_z(y)=\mathbf{s}_0$

$\mu(x)=\mu(y)+1=1$

$HF(S^1 \times S^2, A, \mathbf{s}_0)=$

$A\langle x\rangle \oplus A\langle y\rangle$

Some other simple cases

- Lens spaces L(p,q).

Some other simple cases

- Lens spaces L(p,q).
- $S^3_n(K)$: the result of n-surgery on alternating knots in S^3. The result may be understood in terms of the Alexander polynomial of the knot.

Some other simple cases

- Lens spaces L(p,q).
- $S^3_n(K)$: the result of n-surgery on alternating knots in S^3. The result may be understood in terms of the Alexander polynomial of the knot.
- Connected sums of pieces of the above type: There is a connected sum formula.

Connected sum formula

- $Spin^c(Y_1 \# Y_2) = Spin^c(Y_1) \oplus Spin^c(Y_2)$; Maybe the better notation is $Spin^c(Y_1 \# Y_2) = Spin^c(Y_1) \# Spin^c(Y_2)$

Connected sum formula

- $\mathrm{Spin}^c(Y_1\#Y_2)=\mathrm{Spin}^c(Y_1)\oplus\mathrm{Spin}^c(Y_2)$; Maybe the better notation is $\mathrm{Spin}^c(Y_1\#Y_2)=\mathrm{Spin}^c(Y_1)\#\mathrm{Spin}^c(Y_2)$
- $HF(Y_1\#Y_2,A;\mathbf{s}_1\#\mathbf{s}_2)=$
$$HF(Y_1,A;\mathbf{s}_1)\otimes_A HF(Y_2,A;\mathbf{s}_2)$$

Connected sum formula

- $Spin^c(Y_1\#Y_2)=Spin^c(Y_1)\oplus Spin^c(Y_2)$; Maybe the better notation is $Spin^c(Y_1\#Y_2)=Spin^c(Y_1)\#Spin^c(Y_2)$
- $HF(Y_1\#Y_2,A;\mathbf{s}_1\#\mathbf{s}_2)=$

$$HF(Y_1,A;\mathbf{s}_1)\otimes_A HF(Y_2,A;\mathbf{s}_2)$$

- In particular for A=Z, as a trivial $Z[u_1]$-module, the connected sum formula is usually simple (in practice).

Refinements for knots

- Consider the space of relative $Spin^c$ structures $Spin^c(Y,K)$ for a knot (Y,K);

Refinements for knots

- Consider the space of relative Spin^c structures $\mathrm{Spin}^c(Y,K)$ for a knot (Y,K);
- $\mathrm{Spin}^c(Y,K)$ is by definition the space of homology classes of non-vanishing vector fields in the complement of K which converge to the orientation of K.

Refinements for knots

- The pair of marked points (z,w) on a Heegaard diagram H for K determine a map from the set of generators $\mathbf{x} \in T_\alpha \cap T_\beta$ to $\mathrm{Spin}^c(Y,K)$, denoted by $\mathbf{s}_K(\mathbf{x}) \in \mathrm{Spin}^c(Y,K)$.

Refinements for knots

- The pair of marked points (z,w) on a Heegaard diagram H for K determine a map from the set of generators $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ to Spin^c(Y,K), denoted by $\mathbf{s}_K(\mathbf{x}) \in$ Spin^c(Y,K).
- In the simplest case where A=Z, the coefficient of any $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ in d($\mathbf{x}$) is zero, unless $\mathbf{s}_K(\mathbf{x})=\mathbf{s}_K(\mathbf{y})$.

Refinements for knots

- This is a better refinement in comparison with the previous one for three-manifolds:

  $Spin^c(Y,K)=\mathbf{Z}\oplus Spin^c(Y)$

Refinements for knots

- This is a better refinement in comparison with the previous one for three-manifolds:

  $\mathrm{Spin}^c(Y,K)=\mathbf{Z}\oplus\mathrm{Spin}^c(Y)$

- In particular for $Y=S^3$ and standard knots we have

  $\mathrm{Spin}^c(K):=\mathrm{Spin}^c(S^3,K)=\mathbf{Z}$

We restrict ourselves to this case, with $A=\mathbf{Z}$!

Computations

- HF(K) is completely determined from the symmetrized Alexander polynomial and the signature $\sigma(K)$, if K is an alternating knot.

Computations

- HF(K) is completely determined from the symmetrized Alexander polynomial and the signature $\sigma(K)$, if K is an alternating knot.
- Torus knots, three-strand pretzel knots, etc.

Computations

- HF(K) is completely determined from the symmetrized Alexander polynomial and the signature $\sigma(K)$, if K is an alternating knot.
- Torus knots, three-strand pretzel knots, etc.
- Small knots: We know the answer for all knots up to 14 crossings.

Why is it possible to compute?

- There is an easy way to understand the homotopy classes of disks in $\pi_2(\mathbf{x},\mathbf{y})$ when the associated relative Spin^c structures associated with **x,y** in $\mathrm{Spin}^c(K)$ are the same.

Why is it possible to compute?

- There is an easy way to understand the homotopy classes of disks in $\pi_2(\mathbf{x},\mathbf{y})$ when the associated relative Spin^c structures associated with **x,y** in $\mathrm{Spin}^c(K)$ are the same.
- Let ϕ be an element in $\pi_2(\mathbf{x},\mathbf{y})$, and let $z_1,z_2,\ldots,z_m$ be marked points on S, one in each connected component of the complement of the curves in S.

Why is it possible to compute?

- Consider the subspaces $L(z_j)=\{z_j\}\times Sym^{g-1}(S)$ and let $n(j,\phi)$ be the intersection number of ϕ with $L(z_j)$.

Why is it possible to compute?

- Consider the subspaces $L(z_j)=\{z_j\}\times Sym^{g-1}(S)$ and let $n(j,\phi)$ be the intersection number of ϕ with $L(z_j)$.
- The collection of integers $n(j,\phi)$, $j=1,\ldots,m$ determine the homotopy class ϕ.

Why is it possible to compute?

- Consider the subspaces $L(z_j)=\{z_j\}\times Sym^{g-1}(S)$ and let $n(j,\phi)$ be the intersection number of ϕ with $L(z_j)$.
- The collection of integers $n(j,\phi)$, $j=1,\ldots,m$ determine the homotopy class ϕ.
- There is a simple combinatorial way to check if such a collection determines a homotopy class in $\pi_2(\mathbf{x},\mathbf{y})$ or not.

Why is it possible to compute?

- There is a combinatorial formula for the expected dimension of $\mu(\phi)$ of $M(\phi)$ in terms of $n(j,\phi)$ and the geometry of the curves on S.

Why is it possible to compute?

- There is a combinatorial formula for the expected dimension of $\mu(\phi)$ of $M(\phi)$ in terms of $n(j,\phi)$ and the geometry of the curves on S.
- We know that if $n(\phi)$ is not zero, then $\mu(\phi)=1$, and all $n(j,\phi)$ are non-negative. Furthermore, if $z=z_1$ and $w=z_2$, then $n(1,\phi)= n(2,\phi)=0$.

Why is it possible to compute?

- These are strong restrictions. For example these restrictions are enough for a complete computation for alternating knots.

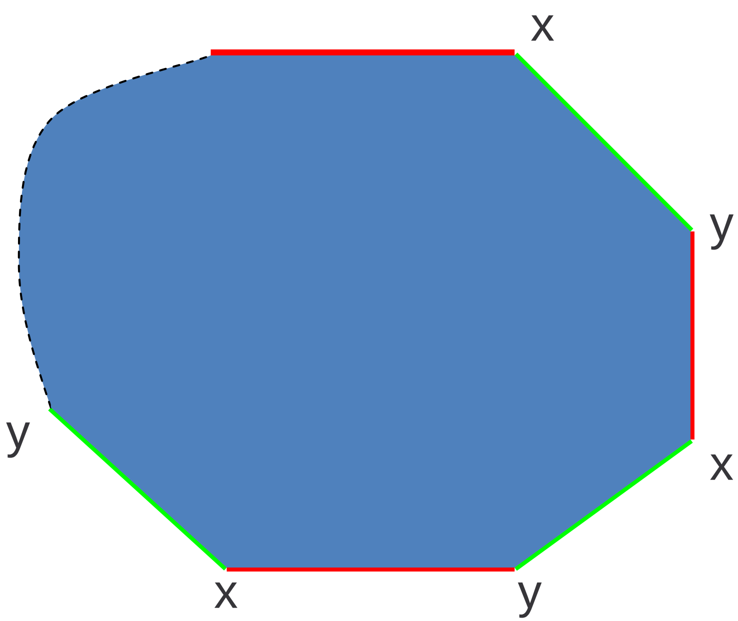
Why is it possible to compute?

- These are strong restrictions. For example these restrictions are enough for a complete computation for alternating knots.
- In other cases, these are still pretty strong, and help a lot with the computations.

Why is it possible to compute?

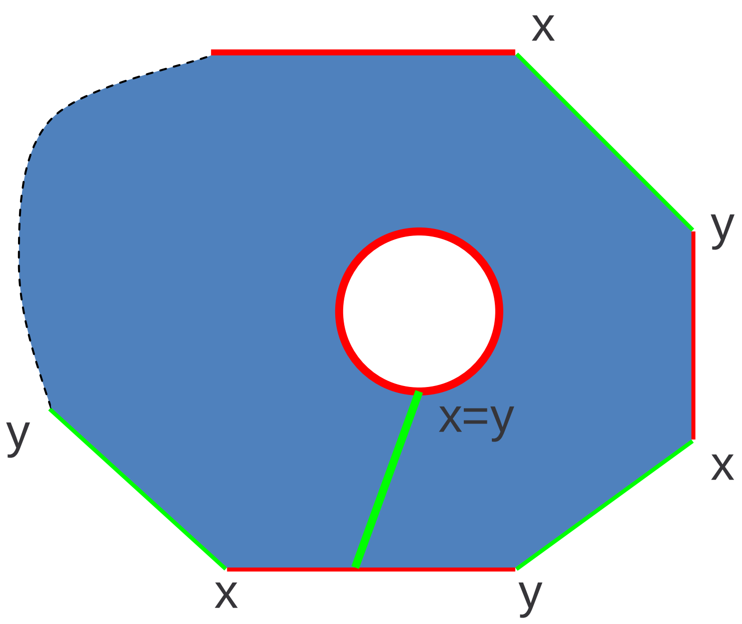
- These are strong restrictions. For example these restrictions are enough for a complete computation for alternating knots.
- In other cases, these are still pretty strong, and help a lot with the computations.
- There are computer programs (e.g. by Monalescue) which provide all the simplifications of the above type in the computations.

Some domains for which the moduli space is known



Any 2n-gone as shown here with alternating red and green edges corresponds to as moduli space contributing 1 to the differential

Some domains for which the moduli space is known



The same is true for the same type of polygons with a number of circles excluded as shown in the picture.

Relation to the three-manifold invariants

- Theorem (Ozsváth-Szabó) Heegaard Floer complex for a knot K determines the Heegaard Floer homology for three-manifolds obtained by surgery on K.

Relation to the three-manifold invariants

- Theorem (E.) More generally if a 3-manifold is obtained from two knot-complements by identifying them on the boundary, then the Heegaard Floer complexs of the two knots, determine the Heegaard Floer homology of the resulting three-manifold