Exterior Differential Systems

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Introduction

In this lecture, I will talk about the theory of exterior differential systems (EDSs) and illustrate on a few very simple examples. Exterior differential systems gives a geometric and coordinate-free approach to the formulation and solution of differential equations, that is, a system of equations on a manifold defined by equating to zero a number of exterior differential forms. In the framework of exterior differential systems, differential equations are replaced by differential ideals in the exterior algebra of differential forms on a manifold, and the solutions of differential equations correspond to integral manifolds of these ideals. Exterior differential systems are thus very well suited to the study of the differential systems that arise in differential geometry and in mechanics, particularly in geometric control theory.

Introduction

Some advantages among which are the facts that the forms themselves often have a geometrical meaning, and that the symmetries of the exterior differential system are larger than those generated simply by changes of independent and dependent variables. Another advantage is that the coordinate-free treatment naturally leads to the intrinsic features of many systems of partial differential equations. Important classical examples of problems that have been treated with great success using exterior differential systems include the local isometric embedding problems in Riemannian geometry, nearly all the classical deformation and classification problems for submanifolds, the local equivalence problem for G-structures, and the study of sub-Riemannian structures and their invariants.

Some Advantages of EDS

- Geometrical meaning
- The larger symmetries than those of the total space of variables
- Coordinate-free treatment and intrinsic features of DEs
- Global properties of geometric objects ...

History of EDS

Pfaff
[1814-15]
Pfaff problems

Frobenius
[1877]
Bilinear covariant

Darbous
[1882]
Bilinear covariant

Goursat
[1890]
Geometrical concept of EDS

Riquier
[1910]
General existance Thm

Elie Cartan
[1922]
ED forms & Derivatives
Maurer-Cartan forms
for infinite Lie pseudo-groups

Kaehler
[1934]
Cartan-Kaehler Thm

Definition

An *exterior differential system* (EDS) is a pair $(M, \mathcal{I})$ where M is a smooth manifold and $\mathcal{I} \subset \Omega^*(M)$ is a graded ideal in the ring $\Omega^*(M)$ of differential forms on M that is closed under exterior differentiation, i.e., for any ϕ in $\mathcal{I}$, its exterior derivative $d\phi$ also lies in $\mathcal{I}$.

The main interest in an EDS $(M, \mathcal{I})$ centers around the problem of describing the submanifolds $f : N \to M$ for which all the elements of $\mathcal{I}$ vanish when pulled back to N, i.e., for which $f^*\phi = 0$ for all $\phi \in \mathcal{I}$. Such submanifolds are said to be *integral manifolds* of $\mathcal{I}$.

In practice, most EDS are constructed so that their integral manifolds will be the solutions of some geometric problem one wants to study. Then the techniques to be described in these lectures can be brought to bear.

Definition

The most common way of specifying an EDS $(M, \mathcal{I})$ is to give a list of generators of $\mathcal{I}$.
For $\phi_1, \ldots, \phi_s \in \Omega^*(M)$, the 'algebraic' ideal consisting of elements of the form

$$\phi = \gamma^1 \wedge \phi_1 + \cdots \gamma^s \wedge \phi_s$$

will be denoted $\langle \phi_1, \ldots, \phi_s \rangle_{\text{alg}}$ while the differential ideal $\mathcal{I}$ consisting of elements of the form

$$\phi = \gamma^1 \wedge \phi_1 + \cdots \gamma^s \wedge \phi_s + \beta^1 \wedge d\phi_1 + \cdots \beta^s \wedge d\phi_s$$

will be denoted $\langle \phi_1, \ldots, \phi_s \rangle$.

Let

$$\mathcal{I}^p = \mathcal{I} \cap \Omega^p(M)$$

$$\Omega^p_x(M) = \Lambda^p(T^*_x M)$$

Example

Élie Cartan developed the theory of exterior differential systems as a coordinate-free way to describe and study partial differential equations. Before I describe the general relationship, let's consider some examples:

Example 1: *An Ordinary Differential Equation.* Consider the system of ordinary differential equations

$$\begin{aligned} y' &= F(x, y, z) \\ z' &= G(x, y, z) \end{aligned}$$

where F and G are smooth functions on some domain $M \subset \mathbb{R}^3$. This can be modeled by the EDS $(M, \mathcal{I})$ where

$$\mathcal{I} = \langle\, dy - F(x, y, z)\, dx,\ dz - G(x, y, z)\, dx \,\rangle.$$

It's clear that the 1-dimensional integral manifolds of $\mathcal{I}$ are just the integral curves of the vector field

$$X = \frac{\partial}{\partial x} + F(x, y, z)\,\frac{\partial}{\partial y} + G(x, y, z)\,\frac{\partial}{\partial z}.$$

Example

Example 2: *A Pair of Partial Differential Equations.* Consider the system of partial differential equations

$$z_x = F(x, y, z)$$
$$z_y = G(x, y, z)$$

where F and G are smooth functions on some domain $M \subset \mathbb{R}^3$. This can be modeled by the EDS $(M, \mathcal{I})$ where

$$\mathcal{I} = \langle\, dz - F(x, y, z)\, dx - G(x, y, z)\, dy \,\rangle.$$

On any 2-dimensional integral manifold $N^2 \subset M$ of $\mathcal{I}$, the differentials dx and dy must be linearly independent . Thus, N can be locally represented as a graph $\big(x, y, u(x, y)\big)$ The 1-form

$$dz - F(x, y, z)\, dx - G(x, y, z)\, dy$$

vanishes when pulled back to such a graph if and only if the function u satisfies the differential equations

$$u_x(x, y) = F\big(x, y, u(x, y)\big)$$
$$u_y(x, y) = G\big(x, y, u(x, y)\big)$$

for all (x, y) in the domain of u.

Example

Example 3: *Complex Curves in* $\mathbb{C}^2$. Consider $M = \mathbb{C}^2$, with coordinates $z = x + i\,y$ and $w = u + i\,v$. Let $\mathcal{I} = \langle \phi_1, \phi_2 \rangle$ where ϕ_1 and ϕ_2 are the real and imaginary parts, respectively, of

$$dz \wedge dw = dx \wedge du - dy \wedge dv + i\,(dx \wedge dv + dy \wedge du).$$

Since $\mathcal{I}^1 = (0)$, any (real) curve in $\mathbb{C}^2$ is an integral curve of $\mathcal{I}$. A (real) surface $N \subset \mathbb{C}^2$ is an integral manifold of $\mathcal{I}$ if and only if it is a complex curve. If dx and dy are linearly independent on N, then locally N can be written as a graph $\bigl(x, y, u(x,y), v(x,y)\bigr)$ where u and v satisfy the Cauchy-Riemann equations: $u_x - v_y = u_y + v_x = 0$. Thus, $(M, \mathcal{I})$ provides a model for the Cauchy-Riemann equations.

Example 4: *Linear Weingarten Surfaces.* This example assumes that you know some differential geometry. Let $M^5 = \mathbb{R}^3 \times S^2$ and let $\mathbf{x} : M \to \mathbb{R}^3$ and $\mathbf{u} : M \to \mathbb{S}^2 \subset \mathbb{R}^3$ be the projections on the two factors. Notice that the isometry group G of Euclidean 3-space acts on M in a natural way, with translations acting only on the first factor and rotations acting 'diagonally' on the two factors together.

Consider the 1-form $\theta = \mathbf{u} \cdot d\mathbf{x}$, which is G-invariant. If $\iota : N \hookrightarrow \mathbb{R}^3$ is an oriented surface, then the lifting $f : N \to M$ given by $f(p) = \bigl(\iota(p), \nu(p)\bigr)$ where $\nu(p) \in S^2$ is the oriented unit normal to the immersion ι at p, is an integral manifold of θ. Conversely, any integral 2-manifold $f : N \to M$ of θ for which the projection $\mathbf{x} \circ f : N \to \mathbb{R}^3$ is an immersion is such a lift of a canonically oriented surface $\iota : N \hookrightarrow \mathbb{R}^3$.

The Frobenius Theorem

Of course, reformulating a system of PDE as an EDS might not necessarily be a useful thing to do. It will be useful if there are techniques available to study the integral manifolds of an EDS that can shed light on the set of integral manifolds and that are not easily applicable to the original PDE system. The main techniques of this type will be discussed in lectures later in the week, but there are a few techniques that are available now.

The first of these is when the ideal $\mathcal{I}$ is algebraically as simple as possible.

Theorem 1: (THE FROBENIUS THEOREM) Let $(M, \mathcal{I})$ be an EDS with the property that $\mathcal{I} = \langle \mathcal{I}^1 \rangle_{\text{alg}}$ and so that $\dim \mathcal{I}^1_p$ is a constant r indepdendent of $p \in M$. Then for each point $p \in M$ there is a coordinate system $\mathbf{x} = (x^1, \ldots, x^{n+r})$ on a p-neighborhood $U \subset M$ so that

$$\mathcal{I}_U = \langle dx^{n+1}, \ldots, dx^{n+r} \rangle.$$

The Frobenius Theorem

In other words, if $\mathcal{I}$ is algebraically generated by 1-forms and has constant 'rank', then $\mathcal{I}$ is locally equivalent to the obvious 'flat' model. In such a case, the n-dimensional integral manifolds of $\mathcal{I}$ are described locally in the coordinate system $\mathbf{x}$ as 'slices' of the form

$$x^{n+1} = c^1, \qquad x^{n+2} = c^2, \qquad \ldots, \qquad x^{n+r} = c^r.$$

In particular, each connected integral manifold of $\mathcal{I}$ lies in a unique maximal integral manifold, which has dimension n. Moreover, these maximal integral manifolds foliate the ambient manifold M.

If you look back at Example 2, you'll notice that $\mathcal{I}$ is generated algebraically by $\mathcal{I}^1$ if and only if it is generated algebraically by

$$\zeta = dz - F(x,y,z)\,dx - G(x,y,z)\,dy,$$

and this, in turn, is true if and only if $\zeta \wedge d\zeta = 0$. (Why?) Now

$$\zeta \wedge d\zeta = \big(F_y - G_x + G\,F_z - F\,G_z\big)\,dx \wedge dy \wedge dz.$$

Thus, by the Frobenius Theorem, if the two functions F and G satisfy the PDE $F_y - G_x + G\,F_z - F\,G_z = 0$, then for every point $(x_0, y_0, z_0) \in M$, there is a function u defined on an open neighborhood of $(x_0, y_0) \in \mathbb{R}^2$ so that $u(x_0, y_0) = z_0$ and so that u satisfies the equations $u_x = F(x, y, u)$ and $u_y = G(x, y, u)$.

The Pfaff Theorem

There is another case (or rather, sequence of cases) in which there is a simple local normal form.

Theorem 2: (THE PFAFF THEOREM) Let $(M, \mathcal{I})$ be an EDS with the property that $\mathcal{I} = \langle \omega \rangle$ for some nonvanishing 1-form ω. Let $r \geq 0$ be the smallest integer for which $\omega{\wedge}(d\omega)^{r+1} \equiv 0$. Then for each point $p \in M$ at which $\omega{\wedge}(d\omega)^r$ is nonzero, there is a coordinate system $\mathbf{x} = (x^1, \ldots, x^{n+2r+1})$ on a p-neighborhood $U \subset M$ so that $\mathcal{I}_U = \langle dx^{n+1} \rangle$ if $r = 0$ and, if $r > 0$, then

$$\mathcal{I}_U = \langle\, dx^{n+1} - x^{n+2}\, dx^{n+3} - x^{n+4}\, dx^{n+5} - \cdots x^{n+2r}\, dx^{n+2r+1} \,\rangle.$$

Note that the case where $r = 0$ is really a special case of the Frobenius Theorem. Points $p \in M$ for which $\omega{\wedge}(d\omega)^r$ is nonzero are known as the *regular* points of the ideal $\mathcal{I}$. The regular points are an open set in M.

The Pfaff Theorem

In fact, the Pfaff Theorem has a slightly stronger form. It turns out that the maximum dimension of an integral manifold of $\mathcal{I}$ that lies in the regular set is $n+r$. Moreover, if $N^{n+r} \subset M$ is such a maximal dimensional integral manifold and N is embedded, then for every $p \in N$, one can choose the coordinates $\mathbf{x}$ so that $N \cap U$ is described by the equations

$$x^{n+1} = x^{n+2} = x^{n+4} = \cdots = x^{n+2r} = 0.$$

In fact, any integral manifold in U near this one on which the $n+r$ functions $x^1, \ldots, x^n, x^{n+3}, x^{n+5}, \ldots, x^{n+2r+1}$ form a coordinate system can be described by equations of the form

$$\begin{aligned} x^{n+1} &= f(x^{n+3}, x^{n+5}, \ldots, x^{n+2r+1}), \\ x^{n+2k} &= \frac{\partial f}{\partial y^k}(x^{n+3}, x^{n+5}, \ldots, x^{n+2r+1}), \quad 1 \leq k \leq r \end{aligned}$$

for some suitable function $f(y^1, \ldots, y^r)$. Thus, one can informally say that the integral manifolds of maximal dimension depend on one arbitrary function of r variables.

Definition

Let G be a Lie group with Lie algebra $\mathfrak{g} = T_eG$, and let η be its canonical left-invariant 1-form. Thus, η is a 1-form on G with values in $\mathfrak{g}$ that satisfies the conditions that, first $\eta_e : T_eG = \mathfrak{g} \to \mathfrak{g}$ is the identity, and, second, that η is left invariant, i.e., $L_a^*(\eta) = \eta$ for all $a \in G$, where $L_a : G \to G$ is left multiplication by a. Let G is a matrix group, with $g : G \to M_n(\mathbb{R})$ the inclusion into the n-by-n matrices, then

$$\eta = g^{-1}\, dg.$$

It is well-known that η satisfies the Maurer-Cartan equation:

$$d\eta = -\tfrac{1}{2}\,[\eta, \eta].$$

The Maurer-Cartan Theorem

Theorem 3: (Maurer-Cartan) If N is connected and simply connected and γ is a smooth $\mathfrak{g}$-valued 1-form on N that satisfies $d\gamma = -\frac{1}{2}[\gamma,\gamma]$, then there exists a smooth map $g : N \to G$, unique up to composition with a constant left translation, so that $g^*\eta = \gamma$.

Sketch of the proof :

Let $M = N \times G$ and consider the $\mathfrak{g}$-valued 1-form

$$\theta = \eta - \gamma.$$

It's easy to compute that

$$d\theta = -\tfrac{1}{2}[\theta,\theta] - [\theta,\gamma].$$

In particular, writing $\theta = \theta^1 x_1 + \cdots + \theta^s x_s$ where $x_1, \ldots, x_s$ is a basis of $\mathfrak{g}$, the differential ideal

$$\mathcal{I} = \langle \theta^1, \ldots, \theta^s \rangle$$

satisfies $\mathcal{I} = \langle \theta^1, \ldots, \theta^s \rangle_{\text{alg}}$. Moreover, the θ^a are manifestly linearly independent since they restrict to each fiber $\{n\} \times G$ to be linearly independent. Thus, the hypotheses of the Frobenius theorem are satisfied, and M is foliated by maximal connected integral manifolds of $\mathcal{I}$, each of which can be shown to project onto the first factor N to be a covering map.

Since N is connected and simply connected, each integral leaf projects diffeomorphically onto N and hence is the graph of a map $g : N \to G$. This g has the desired property.

The Gauss and Codazzi equations

As another typical application of the Frobenius Theorem, I want to consider one of the fundamental theorems of surface theory in Euclidean space.

Let $x : \Sigma \to \mathbb{R}^3$ be an immersion of an oriented surface Σ and let $u : \Sigma \to S^2$ be its Gauss map. In particular $u \cdot dx = 0$. The two quadratic forms

$$\mathrm{I} = dx \cdot dx\,, \qquad\qquad \mathrm{II} = -du \cdot dx$$

are known as the first and second fundamental forms of the oriented immersion x.

It is evident that if $y = Ax + b$ where A lies in $\mathrm{O}(3)$ and b lies in $\mathbb{R}^3$, then y will be an immersion with the same first and second fundamental forms. (NB. The Gauss map of y will be $v = \det(A)\, Au = \pm Au$.) One of the fundamental results of surface theory is a sort of converse to this statement, namely that if $x, y : \Sigma \to \mathbb{R}^3$ have the same first and second fundamental forms, then they differ by an ambient isometry. (Note that the first or second fundamental form alone is not enough to determine the immersion up to rigid motion.) This is known as Bonnet's Theorem, although it appears to have been accepted as true long before Bonnet's proof appeared.

The Gauss and Codazzi equations

The standard argument for Bonnet's Theorem goes as follows: Let $\pi : F \to \Sigma$ be the oriented orthonormal frame bundle of Σ endowed with the metric I. Elements of F consist of triples (p, v_1, v_2) where (v_1, v_2) is an oriented, I-orthonormal basis of $T_p\Sigma$ and $\pi(p, v_1, v_2) = p$. There are unique 1-forms on F, say $\omega_1, \omega_2, \omega_{12}$ so that

$$d\pi(w) = v_1\,\omega_1(w) + v_2\,\omega_2(w)$$

for all $w \in T_{(p,v_1,v_2)}F$ and so that

$$d\omega_1 = -\omega_{12}\wedge\omega_2\,, \qquad\qquad d\omega_2 = \omega_{12}\wedge\omega_1\,.$$

Then

$$\pi^*\mathrm{I} = {\omega_1}^2 + {\omega_2}^2, \qquad\qquad \pi^*\mathrm{I\!I} = h_{11}\,{\omega_1}^2 + 2h_{12}\,\omega_1\omega_2 + h_{22}\,{\omega_2}^2,$$

for some functions h_{11}, h_{12}, and h_{22}. Defining $\omega_{31} = h_{11}\,\omega_1 + h_{12}\,\omega_2$ and $\omega_{32} = h_{12}\,\omega_1 + h_{22}\,\omega_2$, it is not difficult to see that the $\mathbb{R}^3$-valued functions x, $e_1 = x'(v_1)$, $e_2 = x'(v_2)$, and $e_3 = e_1\times e_2$ must satisfy the matrix equation

$$d\begin{bmatrix}1 & 0 & 0 & 0\\ x & e_1 & e_2 & e_3\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\ x & e_1 & e_2 & e_3\end{bmatrix}\begin{pmatrix}0 & 0 & 0 & 0\\ \omega_1 & 0 & \omega_{12} & -\omega_{31}\\ \omega_2 & -\omega_{12} & 0 & -\omega_{32}\\ 0 & \omega_{31} & \omega_{32} & 0\end{pmatrix}.$$

The Gauss and Codazzi equations

$$d\begin{bmatrix} 1 & 0 & 0 & 0 \\ x & e_1 & e_2 & e_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ x & e_1 & e_2 & e_3 \end{bmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ \omega_1 & 0 & \omega_{12} & -\omega_{31} \\ \omega_2 & -\omega_{12} & 0 & -\omega_{32} \\ 0 & \omega_{31} & \omega_{32} & 0 \end{pmatrix}.$$

Now, the matrix

$$\gamma = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \omega_1 & 0 & \omega_{12} & -\omega_{31} \\ \omega_2 & -\omega_{12} & 0 & -\omega_{32} \\ 0 & \omega_{31} & \omega_{32} & 0 \end{pmatrix}$$

takes values in the Lie algebra of the group $G \subset \mathrm{SL}(4, \mathbb{R})$ of matrices of the form

$$\begin{bmatrix} 1 & 0 \\ b & A \end{bmatrix}, \qquad b \in \mathbb{R}^3,\ A \in \mathrm{SO}(3),$$

while the mapping $g : F \to G$ defined by

$$g = \begin{bmatrix} 1 & 0 & 0 & 0 \\ x & e_1 & e_2 & e_3 \end{bmatrix}$$

clearly satisfies $g^{-1}\, dg = \gamma$. Thus, by the uniqueness in Cartan's Theorem, the map g is uniquely determined up to left multiplication by a constant in G.

The Gauss and Codazzi equations

Perhaps more interesting is the application of the existence part of Cartan's Theorem. Given any pair of quadratic forms $(\mathrm{I}, \mathrm{II})$ on a surface Σ with I being positive definite, the construction of F and the accompanying forms $\omega_1, \omega_2, \omega_{12}, \omega_{31}, \omega_{32}$ and thence γ can obviously be carried out. However, it won't necessarily be true that $d\gamma = -\gamma{\wedge}\gamma$. In fact,

$$d\gamma + \gamma \wedge \gamma = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \Omega_{12} & -\Omega_{31} \\ 0 & -\Omega_{12} & 0 & -\Omega_{32} \\ 0 & \Omega_{31} & \Omega_{32} & 0 \end{pmatrix}$$

where, for example,

$$\Omega_{12} = \left(K - h_{11}h_{22} + {h_{12}}^2\right) \omega_1 \wedge \omega_2$$

where K is the Gauss curvature of the metric I. Thus, a necessary condition for the pair $(\mathrm{I}, \mathrm{II})$ to come from an immersion is that the *Gauss equation* hold, i.e.,

$$\boxed{\det\nolimits_{\mathrm{I}} \mathrm{II} = K.}$$

The Gauss and Codazzi equations

The other two expressions $\Omega_{31} = h_1\,\omega_1\wedge\omega_2$ and $\Omega_{32} = h_2\,\omega_1\wedge\omega_2$ are such that there is a well-defined 1-form η on Σ so that $\pi^*\eta = h_1\,\omega_1 + h_2\,\omega_2$. The mapping δ_{I} from quadratic forms to 1-forms that $\mathrm{I\!I} \mapsto \eta$ defines is a first order linear differential operator. Thus, another necessary condition that the pair $(\mathrm{I}, \mathrm{I\!I})$ come from an immersion is that the *Codazzi equation* hold, i.e.,

$$\delta_{\mathrm{I}}(\mathrm{I\!I}) = 0.$$

By Cartan's Theorem, if a pair $(\mathrm{I}, \mathrm{I\!I})$ on a surface Σ satisfy the Gauss and Codazzi equations, then, at least locally, there will exist an immersion $x : \Sigma \to \mathbb{R}^3$ with $(\mathrm{I}, \mathrm{I\!I})$ as its first and second fundamental forms.

Integral Element

Let $(M, \mathcal{I})$ be an EDS. An n-dimensional subspace $E \subset T_x M$ is said to be an *integral element* of $\mathcal{I}$ if

$$\phi(v_1, \ldots, v_n) = 0$$

for all $\phi \in \mathcal{I}^n$ and all $v_1, \ldots, v_n \in E$. The set of all n-dimensional integral elements of $\mathcal{I}$ will be denoted $V_n(\mathcal{I}) \subset G_n(TM)$. $V_n(\mathcal{I})$ is a closed subset of $G_n(TM)$.

Our main interest in integral elements is that the tangent spaces to any n-dimensional integral manifold $N^n \subset M$ are integral elements. Our ultimate goal is to answer the 'converse' questions: When is an integral element tangent to an integral manifold? If so, in 'how many' ways?

It is certainly not always true that every integral element is tangent to an integral manifold.

Example: *Non-existence.* Consider

$$(M, \mathcal{I}) = \big(\mathbb{R}, \langle\, x\, dx \,\rangle\big).$$

The whole tangent space $T_o\mathbb{R}$ is clearly a 1-dimensional integral element of $\mathcal{I}$, but there can't be any 1-dimensional integral manifolds of $\mathcal{I}$.

Extension Space

Let $E \in V_k(\mathcal{I})$ be an integral element and let $(e_1, \ldots, e_k)$ be a basis for $E \subset T_x M$. The set

$$H(E) = \{\, v \in T_x M \;\big|\; \kappa(v, e_1, \ldots, e_k) = 0, \forall \kappa \in \mathcal{I}^{k+1} \} \subseteq T_x M$$

is known as the *polar space* of E, though it probably ought to be called the *extension space* of E, since a vector $v \in T_x M$ lies in $H(E)$ if and only if either it lies in E (the trivial case) or else $E^+ = E + \mathbb{R}v$ lies in $V_{k+1}(\mathcal{I})$. In other words, a $(k+1)$-plane E^+ containing E is an integral element of $\mathcal{I}$ if and only if it lies in $H(E)$.

Now, from the very definition of $H(E)$, it is a vector space and contains E. It is traditional to define the function $r : V_k(\mathcal{I}) \to \{-1, 0, 1, 2, \ldots\}$ by the formula

$$r(E) = \dim H(E) - k - 1.$$

The reason for subtracting 1 is that then $r(E)$ is the dimension of the set of $(k+1)$-dimensional integral elements of $\mathcal{I}$ that contain E, with $r(E) = -1$ meaning that there are no such extensions. When $r(E) \geq 0$, we have

$$\{\, E^+ \in V_{k+1}(\mathcal{I}) \;\big|\; E \subset E^+ \,\} \simeq \mathbb{P}\big(H(E)/E\big) \simeq \mathbb{RP}^{r(E)}.$$

The Cartan-Kahler Theorem

Theorem 4: (Cartan-Kähler) Let $(M, \mathcal{I})$ be a real analytic EDS and suppose that

(1) $P \subset M$ is a connected, k-dimensional, real analytic, regular integral manifold of $\mathcal{I}$ with $r(P) \geq 0$ and

(2) $R \subset M$ is a real analytic submanifold of codimension $r(P)$ containing P and having the property that $T_pR \cap H(T_pP)$ has dimension $k+1$ for all $p \in P$.

There exists a unique, connected, $(k+1)$-dimensional, real analytic integral manifold X of $\mathcal{I}$ that satisfies $P \subset X \subset R$.

The Cauchy-Kowaleski Theorem

Theorem 5: (Cauchy-Kowalewski) Suppose that $D \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^{ns}$ is an open set and suppose that $F : D \to \mathbb{R}^s$ is *real analytic*. Suppose that $U \subset \mathbb{R}^n$ is an open set and that $\phi : U \to \mathbb{R}^s$ is a *real analytic* function with the property that its '1-graph'

$$\left\{ \left(t_0, \mathbf{x}, \phi(\mathbf{x}), \frac{\partial \phi}{\partial \mathbf{x}}(\mathbf{x}) \right) \middle| \, \mathbf{x} \in U \right\}$$

lies in D for some t_0. Then there exists a domain $V \subset \mathbb{R} \times \mathbb{R}^n$ for which $\{t_0\} \times U \subset V$ and a *real analytic* function $\mathbf{u} : V \to \mathbb{R}^s$ satisfying

$$\frac{\partial \mathbf{u}}{\partial t}(t, \mathbf{x}) = F\big(t, \mathbf{x}, \mathbf{u}(t, \mathbf{x}), \frac{\partial \mathbf{u}}{\partial \mathbf{x}}(t, \mathbf{x})\big), \qquad \text{for } (t, \mathbf{x}) \in V$$

$$\mathbf{u}(t_0, \mathbf{x}) = \phi(\mathbf{x}), \qquad \text{for } \mathbf{x} \in U.$$

Moreover, $\mathbf{u}$ is unique as a *real analytic* solution in the sense that any other such $(\tilde{V}, \tilde{\mathbf{u}})$ with $\tilde{\mathbf{u}}$ real analytic satisfies $\tilde{\mathbf{u}} = \mathbf{u}$ on any component of $\tilde{V} \cap V$ that meets $\{t_0\} \times U$.

Integral Flags

A flag of integral elements

$$(0) = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E \subset T_pM$$

where $E_i \in V_i^r(\mathcal{I})$ for $0 \leq i < n$ and $E_n \in V_n(\mathcal{I})$ will be known as a *regular flag* for short. (Note that the terminus E_n of a regular flag is not required to be regular and, in fact, it can fail to be. However, it does turn out that E_n is ordinary.)

Now, corresponding to any integral flag

$$(0) = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E \subset T_pM$$

(regular or not), there is the *descending* flag of corresponding polar spaces

$$T_pM \supseteq H(E_0) \supseteq H(E_1) \supseteq \cdots \supseteq H(E_{n-1}) \supseteq H(E_n) \supseteq E_n\,.$$

In light of the Cartan-Kähler Theorem, there is a simple sufficient condition for the existence of an integral manifold tangent to $E \in V_n(\mathcal{I})$.

Theorem 6: Let $(M, \mathcal{I})$ be a real analytic EDS. If $E \in V_n(\mathcal{I})$ contains a flag of subspaces

$$(0) = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E \subset T_pM$$

where $E_i \in V_i^r(\mathcal{I})$ for $0 \leq i < n$, then there is a real analytic n-dimensional integral manifold $P \subset M$ passing through p and satisfying $T_pP = E$.

Cartan characters

It will be convenient to keep track track of the dimensions of these spaces in terms of their codimension in T_pM. For $k < n$, set

$$c(E_k) = \dim(T_pM) - \dim H(E_k) = n + s - k - 1 - r(E_k)$$

where $\dim M = n+s$. It works out best to make the special convention that $c(E_n) = s$. (In practice, it is usually the case that $H(E_n) = E_n$, in which case, the above formula for $c(E_k)$ works even when you set $k = n$.) Since $\dim H(E_k) \geq \dim E_n = n$, we have $c(E_k) \leq s$. Because of the nesting of these spaces, we have

$$0 \leq c(E_k) \leq c(E_1) \leq \cdots \leq c(E_n) \leq s.$$

For notational convenience, set $c(E_{-1}) = 0$. The *Cartan characters* of the flag $F = (E_0, E_1, \ldots, E_n)$ are the numbers

$$s_k(F) = c(E_k) - c(E_{k-1}) \geq 0.$$

They will play an important role in what follows.

I'm now ready to describe *Cartan's Test*, a necessary and sufficient condition for a given flag to be regular. First, let me introduce some terminology: A subset $X \subset M$ will be said to have *codimension at least q at $x \in X$* if there is an open x-neighborhood $U \subset M$ and a codimension q submanifold $Q \subset U$ so that $X \cap U$ is a subset of Q. In the other direction, X will be said to have *codimension at most q at $x \in X$* if there is an open x-neighborhood $U \subset M$ and a codimension q submanifold $Q \subset U$ containing x so that $Q \subset X \cap U$.

Cartan's Test

Theorem 7: (Cartan's Test) Let $(M, \mathcal{I})$ be an EDS and let $F = (E_0, E_1, \ldots, E_n)$ be an integral flag of $\mathcal{I}$. Then $V_n(\mathcal{I})$ has codmension at least

$$c(F) = c(E_0) + c(E_1) + \cdots + c(E_{n-1})$$

in $G_n(TM)$ at E_n. Moreover, $V_n(\mathcal{I})$ is a smooth submanifold of $G_n(TM)$ of codimension $c(F)$ in a neighborhood of E_n if and only if the flag F is regular.

This is a very powerful result, because it allows one to test for regularity of a flag by simple linear algebra, computing the polar spaces $H(E_k)$ and then checking that $V_n(\mathcal{I})$ is smooth near E_n and of the smallest possible codimension, $c(F)$. In many cases, these two things can done by inspection.

Example

Example: *Self-Dual 2-Forms.* Any integral element $E \in V_4(\mathcal{I}) \cap G_4(T\mathbb{R}^7, d\mathbf{x})$ is defined by linear equations of the form

$$\pi^a = du^a - p_i^a(E)\,dx^i = 0.$$

In order that Φ vanish on such a 4-plane, it suffices that the $p_i^a(E)$ satisfy four equations:

$$p_1^1 + p_2^2 + p_3^3 = p_4^1 - p_3^2 + p_2^3 = p_4^2 - p_1^3 + p_3^1 = p_4^3 - p_2^1 + p_1^2 = 0$$

It's clear from this that $V_4(\mathcal{I}) \cap G_4(T\mathbb{R}^7, d\mathbf{x})$ is a smooth manifold of codimension 4 in $G_4(T\mathbb{R}^7)$. On the other hand, if we let $E_k \subset E$ be defined by the equation $dx^{k+1} = dx^{k+2} = \cdots = dx^4 = 0$ for $0 \le k < 4$, then it is easy to see that

$$\begin{aligned} H(E_0) = H(E_1) &= T_p(M) \\ H(E_2) &= \{ v \in T_p(M) \mid \pi_3(v) = 0 \} \\ H(E_3) &= \{ v \in T_p(M) \mid \pi_1(v) = \pi_2(v) = \pi_3(v) = 0 \} \\ H(E_4) &= \{ v \in T_p(M) \mid \pi_1(v) = \pi_2(v) = \pi_3(v) = 0 \} \end{aligned}$$

so $c(E_0) = c(E_1) = 0$, $c(E_2) = 1$, $c(E_3) = 3$, and $c(E_4) = 3$. Since $c(F) = 0+0+1+3 = 4$, which is the codimension of $V_4(\mathcal{I})$ in $G_4(T\mathbb{R}^7)$, Cartan's Test is verified and the flag is regular.

Weingarten Surface

Let $x : N \to \mathbb{R}^3$ be an immersion of an oriented surface and let $u : N \to S^2$ be the associated oriented normal, sometimes known as the Gauss map. Recall that we have the two fundamental forms

$$\mathrm{I} = dx \cdot dx, \qquad \mathrm{II} = -du \cdot dx.$$

The eigenvalues of II with respect to I are known as the *principal curvatures* of the immersion. On the open set $N^* \subset N$ where the two eigenvalues are distinct, they are smooth functions on N. The complement $N \setminus N^*$ is known as the *umbilic locus*. For simplicity, I am going to suppose that $N^* = N$, though many of the constructions that I will do can, with some work, be made to go through even in the presence of umbilics.

Possibly after passing to a double cover, we can define vector-valued functions $e_1, e_2 : N \to \mathbb{S}^2$ so that $e_1 \times e_2 = u$ and so that, setting $\eta^i = e_i \cdot dx$, we can write

$$\begin{aligned} dx &= e_1 \quad \eta_1 + e_2 \quad \eta_2 \,, \\ -du &= e_1 \ \kappa_1 \, \eta_1 + e_2 \ \kappa_2 \, \eta_2 \,, \end{aligned}$$

where $\kappa_1 > \kappa_2$ are the principal curvatures. The immersion x defines a *Weingarten surface* if the principal curvatures satisfy a (non-trivial) relation of the form $F(\kappa_1, \kappa_2) = 0$. (For a generic immersion, the functions κ_i satisfy $d\kappa_1 \wedge d\kappa_2 \neq 0$, at least on a dense open set.) For example, the equations $\kappa_1 + \kappa_2 = 0$ and $\kappa_1 \kappa_2 = 1$ define Weingarten relations, perhaps better known as the relations $H = 0$ (minimal surfaces) and $K = 1$, respectively.

The Involutivity Condition

Let $(M, \mathcal{I})$ be an EDS and let $Z \subset V_n^o(\mathcal{I})$ be a connected open subset of $V_n^o(\mathcal{I})$. We say that Z is *involutive* if every $E \in Z$ is the terminus of a regular flag. Usually, in applications, there is only one such Z to worry about anyway, or else the 'interesting' component Z is clear from context, in which case we simply say that $(M, \mathcal{I})$ is involutive.

The first piece of good news about the prolongation process is that it doesn't destroy involutivity:

Theorem 8: (Persistence of Involutivity) Let $(M, \mathcal{I})$ be an EDS with $\mathcal{I}^0 = (0)$ and let $M^{(1)} \subset V_n^o(\mathcal{I})$ be a connected open subset of $V_n^o(\mathcal{I})$ that is involutive. Then the character sequence $(s_0(F), \ldots, s_n(F))$ is the same for all regular flags $F = (E_0, \ldots, E_n)$ with $E_n \in M^{(1)}$. Moreover, the EDS $(M^{(1)}, \mathcal{I}^{(1)})$ is involutive on the set $M^{(2)} \subset V_n(\mathcal{I}^{(1)})$ of elements that are transverse to the projection $\pi : M^{(1)} \to M$ and its character sequence $(s_0^{(1)}, \ldots, s_n^{(1)})$ is given by

$$s_k^{(1)} = s_k + s_{k+1} + \cdots + s_n \,.$$

The Cartan-Kuranishi Theorem

Theorem 9: (Cartan-Kuranishi) Suppose that one has a sequence of manifolds M_k for $k \geq 0$ together with embeddings $\iota_k : M_k \hookrightarrow G_n(TM_{k-1})$ for $k > 0$ with the properties

(1) The composition $\pi_{k-1} \circ \iota_k : M_k \to M_{k-1}$ is a submersion,

(2) For all $k \geq 2$, $\iota_k(M_k)$ is a submanifold of $V_n(\mathcal{C}_{k-2}, \pi_{k-2})$, the integral elements of the contact system $\mathcal{C}_{k-2}$ on $G_n(TM_{k-2})$ transverse to the fibers of $\pi_{k-2} : G_n(TM_{k-2}) \to M_{k-2}$.

Then there exists a $k_0 \geq 0$ so that for $k \geq k_0$, the submanifold $\iota_{k+1}(M_{k+1})$ is an involutive open subset of $V_n\big(\iota_k^* \mathcal{C}_{k-1}\big)$, where $\iota_k^* \mathcal{C}_{k-1}$ is the EDS on M_k pulled back from $G_n(TM_{k-1})$.

Prolongation

A sequence of manifolds and immersions as described in the theorem is sometimes known as a *prolongation sequence*.

Now, you can imagine how this theorem might be useful. When you start with an EDS $(M, \mathcal{I})$ and some submanifold $\iota : Z \hookrightarrow V_n(\mathcal{I})$ that is not involutive, you can start building a prolongation sequence by setting $M_1 = Z$ and looking for a submanifold $M_2 \subset V_n(\iota^*\mathcal{C}_0)$ that is some component of $V_n(\iota^*\mathcal{C}_0)$. You keep repeating this process until either you get to a stage M_k where $V_n(\iota^*\mathcal{C}_{k-1})$ is empty, in which case there aren't any integral manifolds of this kind, or else, eventually, this will have to result in an involutive system, in which case you can apply the Cartan-Kähler Theorem (if the system that you started with is real analytic).

The main difficulty that you'll run into is that the spaces $V_n(\mathcal{I})$ can be quite wild and hard to describe. I don't want to dismiss this as a trivial problem, but it really is an algebra problem, in a sense. The other difficulty is that the the components $M_1 \subset V_n(\mathcal{I})$ might not submerse onto $M_0 = M$, but onto some proper submanifold, in which case, you'll have to restrict to that submanifold and start over.

In the case that the original EDS $(M, \mathcal{I})$ is real analytic, the set $V_n(\mathcal{I}) \subset G_n(TM)$ will also be real analytic and so has a canonical stratification into submanifolds

$$V_n(\mathcal{I}) = \bigcup_{\beta \in B} Z_\beta\,.$$

One can then consider the family of prolongations $(Z_\beta, \mathcal{I}_\beta^{(1)})$ and analyse each one separately. (Fortunately, in all the interesting cases I'm aware of, the number of strata is mercifully small.)

Prolongation

Now, there are precise, though somewhat technical, hypotheses that will ensure that this *prolongation Ansatz*, when iterated and followed down all of its various branches, terminates after a finite number of steps, with the result being a finite (possibly empty) set of EDSs $\{ (M_\gamma, \mathcal{I}_\gamma) \,|\, \gamma \in \Gamma \}$ that are involutive. This result (with the explicit technical hypotheses) is due to Kuranishi and is known as the Cartan-Kuranishi Prolongation Theorem. (Cartan had conjectured/stated this result in his earlier writings, but never provided adequate justification for his claims.) In practice, though, Kuranishi's result is used more as a justification for carrying out the process of prolongation as part of the analysis of an EDS, when it is necessary.

Isometric Embedding of
surfaces with Prescribed Mean Curvature

Consider a given abstract oriented surface N^2 endowed with a Riemannian metric g and a choice of a smooth function H. The question we ask is this: When does there exist an isometric embedding $x : N^2 \to \mathbb{R}^3$ such that the mean curvature function of the immersion is H? If you think about it, this is four equations for the map x (which has three components), three of first order (the isometric embedding condition) and one of second order (the mean curvature restriction).

Since $H^2 - K = (\kappa_1 - \kappa_2)^2 \geq 0$ for any surface in 3-space, one obvious restriction coming from the Gauss equation is that $H^2 - K$ must be nonnegative, where K is the Gauss curvature of the metric g. I'm just going to treat the case where $H^2 - K$ is strictly positive, though there are methods for dealing with the 'umbilic locus' (I just don't want to bother with them here). In fact, set $r = \sqrt{H^2 - K} > 0$.

The simplest way to set up the problem is to begin by fixing an oriented, g-orthonormal coframing (η_1, η_2), with dual frame field (u_1, u_2). We know that there exists a unique 1-form η_{12} so that

$$d\eta_1 = -\eta_{12} \wedge \eta_2, \qquad d\eta_2 = \eta_{12} \wedge \eta_1, \qquad d\eta_{12} = K\, \eta_1 \wedge \eta_2 .$$

Isometric Embedding of
surfaces with Prescribed Mean Curvature

This suggests setting up the following exterior differential system for the 'graph' of f in $N \times F$. Let $M = N \times F \times S^1$, with ϕ being the 'coordinate' on the S^1 factor and consider the ideal $\mathcal{I}$ generated by the five 1-forms

$$\begin{aligned}
\theta_0 &= \omega_3 \\
\theta_1 &= \omega_1 - \eta_1 \\
\theta_2 &= \omega_2 - \eta_2 \\
\theta_3 &= \omega_{12} - \eta_{12} \\
\theta_4 &= \omega_{31} - (H + r\cos\phi)\,\eta_1 - r\sin\phi\,\eta_2 \\
\theta_5 &= \omega_{32} - r\sin\phi\,\eta_1 - (H - r\cos\phi)\,\eta_2
\end{aligned}$$

It's easy to see (and you should check) that

$$d\theta_0 \equiv d\theta_1 \equiv d\theta_2 \equiv d\theta_3 \equiv 0 \mod \{\theta_0, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5\}.$$

Isometric Embedding of
surfaces with Prescribed Mean Curvature

The interesting case will come when we look at the other two 1-forms. In fact, the formula for these is simply

$$\left.\begin{aligned} d\theta_4 &\equiv \quad r\tau \wedge (\sin\phi\,\eta_1 - \cos\phi\,\eta_2) \\ d\theta_5 &\equiv -r\tau \wedge (\cos\phi\,\eta_1 + \sin\phi\,\eta_2) \end{aligned}\right\} \bmod \{\theta_0, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5\}$$

where, setting $dr = r_1\,\eta_1 + r_2\,\eta_2$ and $dH = H_1\,\eta_1 + H_2\,\eta_2$,

$$\begin{aligned} \tau = d\phi - 2\,\eta_{12} - r^{-1}(r_2 + H_2\cos\phi - H_1\sin\phi)\,\eta_1 \\ + r^{-1}(r_1 - H_1\cos\phi - H_2\sin\phi)\,\eta_2\,. \end{aligned}$$

It is clear that there is a unique integral element at each point of M and that it is described by $\theta_0 = \cdots = \theta_5 = \tau = 0$. Thus, $M^{(1)} = M$ and

$$\mathcal{I}^{(1)} = \langle\, \theta_0, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \tau\,\rangle.$$

Isometric Embedding of
surfaces with Prescribed Mean Curvature

To get the structure of $\mathcal{I}^{(1)}$ is is only necessary to compute $d\tau$ now and the result of that is

$$d\tau \equiv r^{-2}\big(C\,\cos\phi + S\,\sin\phi + T\big)\,\eta_1\wedge\eta_2 \mod \{\theta_0,\theta_1,\theta_2,\theta_3,\theta_4,\theta_5,\tau\}$$

where the functions C, S, and T are defined on the surface by

$$\begin{aligned} C &= 2r_1H_1 - 2r_2H_2 - rH_{11} + rH_{22}\,,\\ S &= 2r_2H_1 + 2r_1H_2 - 2rH_{12}\,,\\ T &= 2r^4 - 2H^2r^2 + r(r_{11}+r_{22}) - {r_1}^2 - {r_2}^2 - {H_1}^2 - {H_2}^2\,. \end{aligned}$$

and I have defined H_{ij} and r_{ij} by the equations

$$\begin{aligned} dH_1 &= -H_2\,\eta_{12} + H_{11}\,\eta_1 + H_{12}\,\eta_2\,,\\ dH_2 &= H_1\,\eta_{12} + H_{12}\,\eta_1 + H_{22}\,\eta_2\,,\\ dr_1 &= -r_2\,\eta_{12} + r_{11}\,\eta_1 + r_{12}\,\eta_2\,,\\ dr_2 &= r_1\,\eta_{12} + r_{12}\,\eta_1 + r_{22}\,\eta_2\,. \end{aligned}$$

Isometric Embedding of
surfaces with Prescribed Mean Curvature

Clearly, there are no integral elements of $\mathcal{I}^{(1)}$ except along the locus where $C\cos\phi + S\sin\phi + T = 0$, so it's a question of what this locus looks like.

First, off, note that if $T^2 > S^2 + C^2$, then this locus is empty. Now, this inequality is easily seen not to depend on the choice of coframing (η_1, η_2) that we made to begin with. It depends only on the metric g and the function H. One way to think of this is that the condition $T^2 \leq S^2 + C^2$ is a differential inequality any g and H satisfy if they are the metric and mean curvature of a surface in $\mathbb{R}^3$.

Now, when $T^2 < C^2 + S^2$, there will be exactly two values of ϕ (mod 2π) that satisfy $C\cos\phi + S\sin\phi + T = 0$, say ϕ_+ and ϕ_-, thought of as functions on the surface N. If you restrict to this double cover $\phi = \phi_\pm$, we now have an ideal $\mathcal{I}^{(1)}$ on an 8-manifold that is generated by seven 1-forms. In fact, $\theta_0, \ldots, \theta_5$ are clearly independent, but now

$$\tau = E_1\,\eta_1 + E_2\,\eta_2$$

where E_1 and E_2 are functions on the surface $\tilde{N} \subset N \times S^1$ defined by the equation $C\cos\phi + S\sin\phi + T = 0$. Wherever either of these functions is nonzero, there is clearly no solution. On the other hand, if $E_1 = E_2 = 0$ on $\tilde{N}$, then there are exactly two geometrically distinct ways for the surface to be isometrically embeded with mean curvature H. If you unravel this, you will see that it is a pair of fifth order equations on the pair (g, H). (The expressions T and $S^2 + C^2$ are fourth order in g and second order in H.)

Another possibility is that $T = C = S = 0$, in which case $\mathcal{I}^{(1)}$ becomes Frobenius.

Isometric Embedding of
surfaces with Prescribed Mean Curvature

Of course, this raises the question of whether there exist any pairs (g, H) satisfying these equations. One way to try to satisfy the equations is to look for special solutions. For example, if H were constant, then H_1, H_2, H_{11}, H_{12}, and H_{22} would all be zero, of course, so this would automatically make $C = S = 0$ and then there is only one more equation to satisfy, which can now be reëxpressed, using $K = H^2 - r^2$, as

$$T = r^2\big(\Delta_g \ln(H^2 - K) - 4K\big) = 0$$

where Δ_g is the Laplacian associated to g.

It follows that any metric g on a simply connected surface N that satisfies the fourth order differential equation $\Delta_g \ln(H^2 - K) - 4K = 0$ can be isommetrically embedded in $\mathbb{R}^3$ as a surface of constant mean curvature H in a 1-parameter family (in fact, an S^1) of ways. In particular, we have Bonnet's Theorem:

Any simply connected surface in $\mathbb{R}^3$ with constant mean curvature can be isometrically deformed in an circle of ways preserving the constant mean curvature.

Isometric Embedding of
surfaces with Prescribed Mean Curvature

However, the cases where H is constant give only one special class of solutions of the three equations $C = S = T = 0$. Could there be others?

Well, let's restrict to the open set $U \subset N$ where $dH \neq 0$, i.e., where ${H_1}^2 + {H_2}^2 > 0$. Remember, the original coframing (η_1, η_2) we chose was arbitrary, so we might as well use the nonconstancy of H to tack this down. In fact, let's take our coframing so that the dual frame field (u_1, u_2) has the property that u_1 points in the direction of steepest increase for H, i.e., in the direction of the gradient of H. This means that, for this coframing $H_2 = 0$ and $H_1 > 0$.

The equations $C = S = 0$ now simplify to

$$H_{12} = (r_2/r)\, H_1, \qquad\qquad H_{11} - H_{22} = (2r_1/r)H_1\,.$$

Moreover, looking back at the structure equations found so far, this implies that $dH = H_1\,\eta_1$ and that there is a function P so that

$$\begin{aligned} {H_1}^{-1} dH_1 &= (rP + r_1/r)\,\eta_1 + (r_2/r)\,\eta_2, \,, \\ -\eta_{12} &= (r_2/r)\,\eta_1 + (rP - r_1/r)\,\eta_2\,. \end{aligned}$$

Isometric Embedding of
surfaces with Prescribed Mean Curvature

The first equation can be written in the form

$$d\big(\ln(H_1/r)\big) = r P\,\eta_1.$$

Differentiating this and using the structure equations we have so far then yields that $dP\wedge\eta_1 = 0$, so that there is some λ so that $dP = \lambda\,\eta_1$. On the other hand, differentiating the second of the two equations above and using $T = 0$ to simplify the result, we see that the multiplier λ is determined. In fact, we must have

$$dP = (r^2H^2 + {H_1}^2 - r^4 - r^4P^2)\,\eta_1\,.$$

Differentiating this relation and using the equations we have found so far yields

$$0 = 2r^{-4}\big({H_1}^2 + r^2H^2\big)\,r_2\,\eta_1\wedge\eta_2\,.$$

In particular, we must have $r_2 = 0$. Of course, this simplifies the equations even further. Taking the components of $0 = dr_2 = r_1\eta_{12} + r_{11}\,\eta_1 + r_{22}\,\eta_2$ together with the equation $T = 0$ allows us to solve for r_{11}, r_{12}, and r_{22} in terms of $\{r, H, r_1, H_1, P\}$.

Isometric Embedding of
surfaces with Prescribed Mean Curvature

In fact, collecting all of this information, we get the following structure equations for any solution of our problem:

$$
\begin{aligned}
d\eta_1 &= 0\\
d\eta_2 &= (rP - r_1/r)\,\eta_1 \wedge \eta_2\\
dr &= r_1\,\eta_1\\
dH &= H_1\,\eta_1\\
dr_1 &= (2r^3 - 2H^2 r + r_1 r P - 2{r_1}^2/r - {H_1}^2/r)\,\eta_1\\
dH_1 &= H_1(rP + r_1/r)\,\eta_1\\
dP &= (r^2H^2 + {H_1}^2 - r^4 - r^4P^2)\,\eta_1
\end{aligned}
$$

These may not look promising, but, in fact, they give a rather complete description of the pairs (g, H) that we are seeking. Suppose that N is simply connected. The first structure equation then says that $\eta_1 = dx$ for some function x, uniquely defined up to an additive constant. The last 5 structure equations then say that the functions (r, H, r_1, H_1, P) are solutions of the ordinary differential equation system

$$
\begin{aligned}
r' &= r_1\\
H' &= H_1\\
r_1' &= (2r^3 - 2H^2 r + r_1 r P - 2{r_1}^2/r - {H_1}^2/r)\\
H_1' &= H_1(rP + r_1/r)\\
P' &= (r^2H^2 + {H_1}^2 - r^4 - r^4P^2)
\end{aligned}
$$

Isometric Embedding of
surfaces with Prescribed Mean Curvature

Obviously, this defines a vector field on the open set in $\mathbb{R}^5$ defined by $r > 0$, and there is a four parameter family of integral curves of this vector field. Given a solution of this ODE system on some maximal x-interval, there will be a function F uniquely defined up to an additive constant so that

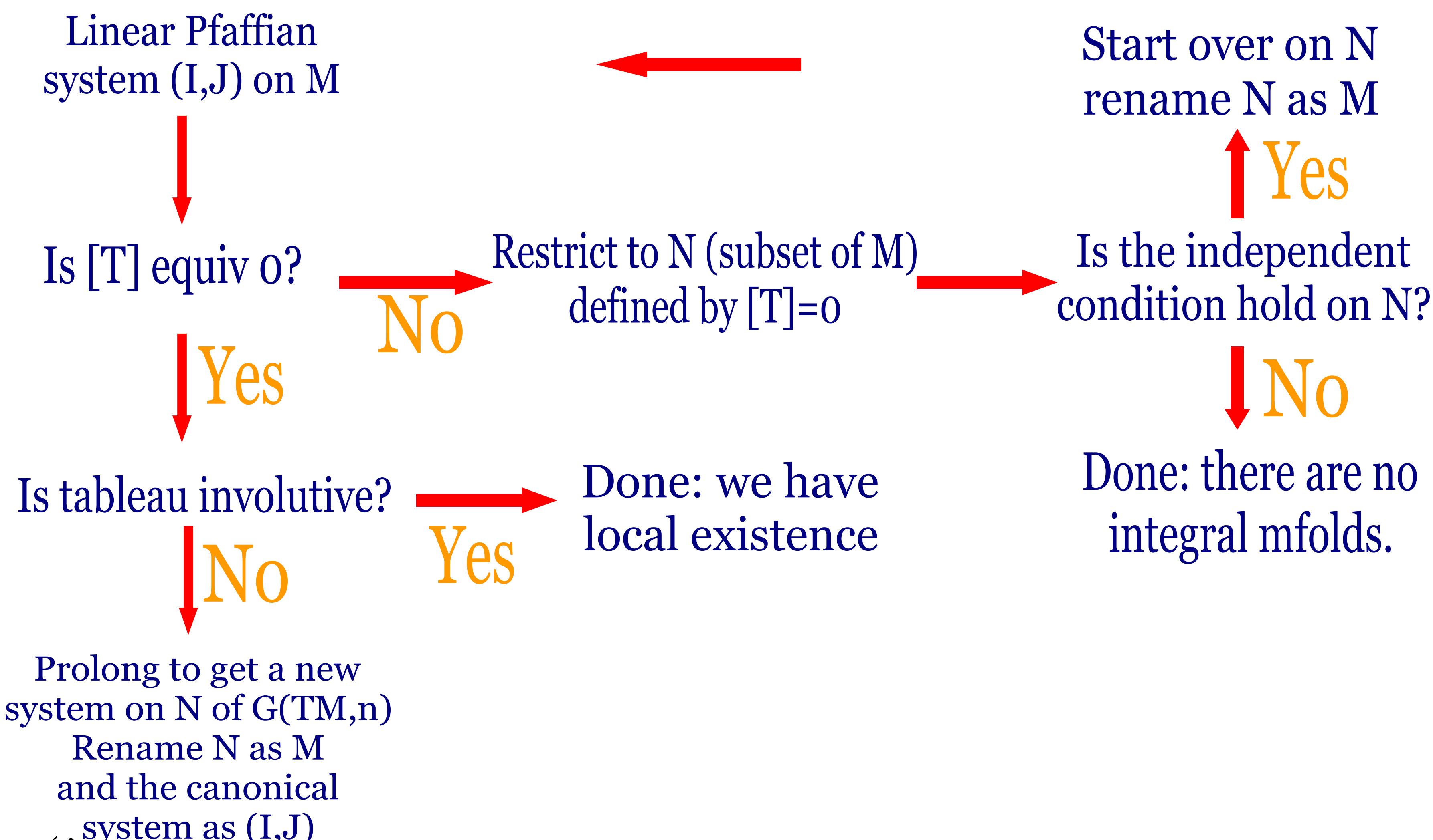
$$F' = (rP - r_1/r).$$

Now by the second structure equation, we have $d(e^{-F}\eta_2) = 0$, so that there must exist a function y on the surface N so that $\eta_2 = e^F\,dy$. Thus, in the (x, y)-coordinates, the metric is of the form

$$g = dx^2 + e^{2F(x)}\,dy^2$$

where (r, H, r_1, H_1, P, F) satisfy the above equations.

EDS Algorithm




The End
My Sincerely Thanks
for your attentions