Which graph properties are characterized by the spectrum?

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Celebrating 80 years

Reza Khosrovshahi

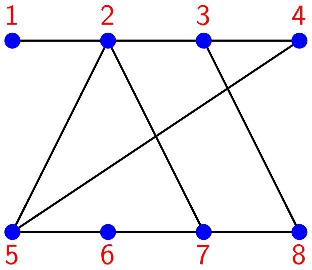

LEKÁREŇ
CENTRUM U ZLATÉHO GRÍFA
PREDPREDAJ


MAINE

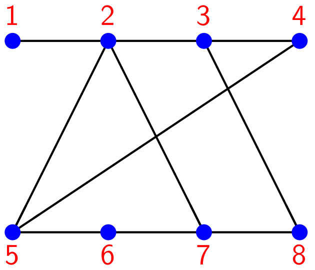

Bos

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$



$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$



adjacency spectrum

$$\{-1-\sqrt{3}\,,\ \ -1\,,\ \ -1\,,\ \ 1-\sqrt{3}\,,\ \ -1+\sqrt{3}\,,\ \ 1\,,\ \ 1\,,\ \ 1+\sqrt{3}\}$$

adjacency spectrum

$$\{-1-\sqrt{3}\,,\;\; -1\,,\;\; -1\,,\;\; 1-\sqrt{3}\,,\;\; -1+\sqrt{3}\,,\;\; 1\,,\;\; 1\,,\;\; 1+\sqrt{3}\}$$

adjacency spectrum

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The adjacency spectrum is symmetric around 0

adjacency spectrum

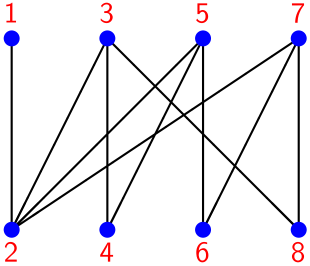
$$\{-1-\sqrt{3}\,,\ \ -1\,,\ \ -1\,,\ \ 1-\sqrt{3}\,,\ \ -1+\sqrt{3}\,,\ \ 1\,,\ \ 1\,,\ \ 1+\sqrt{3}\}$$

Theorem (Coulson, Rushbrooke 1940, Sachs 1966)

The adjacency spectrum is symmetric around 0

if and only if the graph is bipartite

$$
\begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0
\end{bmatrix}
$$



adjacency spectrum

$$\{-1-\sqrt{3}\ ,\ \ -1\ ,\ \ -1\ ,\ \ 1-\sqrt{3}\ ,\ \ -1+\sqrt{3}\ ,\ \ 1\ ,\ \ 1\ ,\ \ 1+\sqrt{3}\}$$

$\lambda_1 \geq \ldots \geq \lambda_n$ are the adjacency eigenvalues of G

Theorem

G has n vertices, $\frac{1}{2}\sum_{i=1}^{n} \lambda_i^2$ edges and $\frac{1}{6}\sum_{i=1}^{n} \lambda_i^3$ triangles

Theorem

G is regular if and only if λ_1 equals the average degree

$\lambda_1 \geq \ldots \geq \lambda_n$ are the adjacency eigenvalues of G

Theorem

G has n vertices, $\frac{1}{2}\sum_{i=1}^{n} \lambda_i^2$ edges and $\frac{1}{6}\sum_{i=1}^{n} \lambda_i^3$ triangles

Theorem

G is regular if and only if λ_1 equals $\frac{1}{n}\sum_{i=1}^{n} \lambda_i^2$

$\lambda_1 \geq \ldots \geq \lambda_n$ are the adjacency eigenvalues of G

Theorem

G has n vertices, $\frac{1}{2}\sum_{i=1}^{n} \lambda_i^2$ edges and $\frac{1}{6}\sum_{i=1}^{n} \lambda_i^3$ triangles

Theorem

G is regular if and only if λ_1 equals $\frac{1}{n}\sum_{i=1}^{n} \lambda_i^2$

Drawback

Spectrum does not tell everything

$$\{-1-\sqrt{3}\ ,\ \ -1\ ,\ \ -1\ ,\ \ 1-\sqrt{3}\ ,\ \ -1+\sqrt{3}\ ,\ \ 1\ ,\ \ 1\ ,\ \ 1+\sqrt{3}\}$$

$$\{-1-\sqrt{3}\ ,\ \ -1\ ,\ \ -1\ ,\ \ 1-\sqrt{3}\ ,\ \ -1+\sqrt{3}\ ,\ \ 1\ ,\ \ 1\ ,\ \ 1+\sqrt{3}\}$$

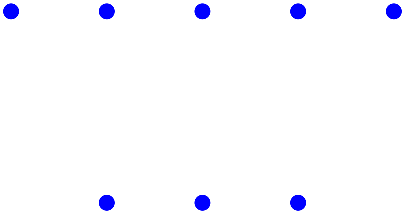
8 vertices, 10 edges, bipartite

$$\{-1-\sqrt{3}\,,\ \ -1\,,\ \ -1\,,\ \ 1-\sqrt{3}\,,\ \ -1+\sqrt{3}\,,\ \ 1\,,\ \ 1\,,\ \ 1+\sqrt{3}\}$$

8 vertices, 10 edges, bipartite

Can the bipartition have parts of unequal size?

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & & & \\ & & & & & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 \end{bmatrix}$$

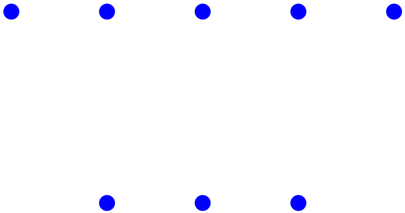


$$\{-1-\sqrt{3}\ ,\ \ -1\ ,\ \ -1\ ,\ \ 1-\sqrt{3}\ ,\ \ -1+\sqrt{3}\ ,\ \ 1\ ,\ \ 1\ ,\ \ 1+\sqrt{3}\}$$

8 vertices, 10 edges, bipartite

Can the bipartition have parts of unequal size? NO!

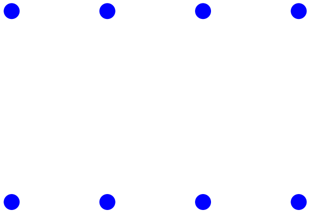
$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & & & \\ & & & & & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 \end{bmatrix}$$



$$\{-1-\sqrt{3}\ ,\quad -1\ ,\quad -1\ ,\quad 1-\sqrt{3}\ ,\quad -1+\sqrt{3}\ ,\quad 1\ ,\quad 1\ ,\quad 1+\sqrt{3}\}$$

8 vertices, 10 edges, bipartite with parts of size 4

$$\begin{bmatrix} 0 & 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & & & & \\ & & & & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \end{bmatrix}$$



$$\{-1-\sqrt{3}\ ,\ \ -1\ ,\ \ -1\ ,\ \ 1-\sqrt{3}\ ,\ \ -1+\sqrt{3}\ ,\ \ 1\ ,\ \ 1\ ,\ \ 1+\sqrt{3}\}$$

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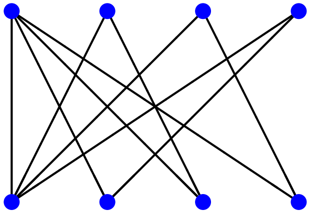
$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

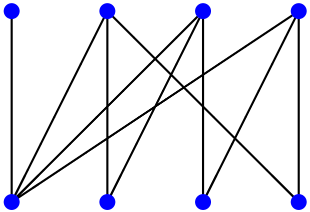


degree sequence $(2, 2, 2, 2, 2, 2, 4, 4)$

$$\{-1-\sqrt{3}\ ,\ \ -1\ ,\ \ -1\ ,\ \ 1-\sqrt{3}\ ,\ \ -1+\sqrt{3}\ ,\ \ 1\ ,\ \ 1\ ,\ \ 1+\sqrt{3}\}$$

8 vertices, 10 edges, bipartite with parts of size 4

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$



degree sequence $(1, 2, 2, 2, 3, 3, 3, 4)$

Observation

The degree sequence of a graph is not determined by the adjacency spectrum

Question

Are the sizes of the two parts of a bipartite graph determined by the adjacency spectrum?

Observation

The degree sequence of a graph is not determined by the adjacency spectrum

Question

Are the sizes of the two parts of a bipartite graph determined by the adjacency spectrum?

General answer is NO!

both graphs have adjacency spectrum

$$\{-2,\ 0,\ 0,\ 0,\ 2\}$$

Problem (Zwierzyński 2006)

Can one determine the size of a bipartition given only the spectrum of a connected bipartite graph?

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Can one determine the size of a bipartition given only the spectrum of a connected bipartite graph?

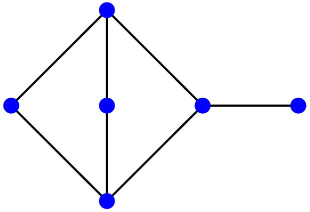
Theorem (van Dam, WHH 2008)

NO!

NOT determined by the adjacency spectrum are:

- being connected
- being a tree
- the girth

Laplacian (matrix)



$$\begin{bmatrix} 3 & -1 & -1 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & -1 \\ -1 & 0 & 2 & 0 & 0 & -1 \\ -1 & 0 & 0 & 3 & -1 & -1 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 3 \end{bmatrix}$$

Laplacian spectrum

$$\{0,\ 3-\sqrt{5},\ 2,\ 3,\ 3,\ 3+\sqrt{5}\}$$

$0 = \mu_1 \leq \ldots \leq \mu_n$ are the Laplacian eigenvalues of G

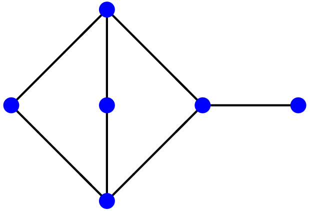
Theorem

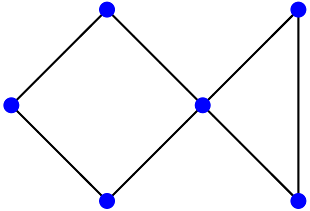
- G has $\frac{1}{2}\sum_{i=2}^{n} \mu_i$ edges, and $\frac{1}{n}\prod_{i=2}^{n} \mu_i$ spanning trees

- the number of connected components of G equals the multiplicity of 0

Theorem

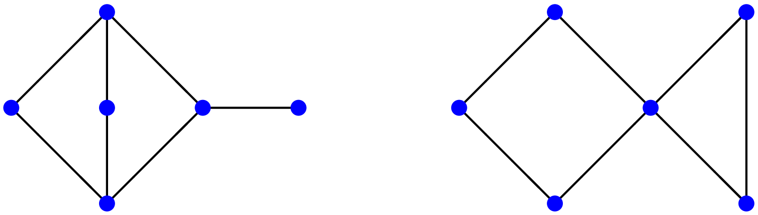
G is regular if and only if $n\sum_{i=2}^{n} \mu_i(\mu_i - 1) = (\sum_{i=2}^{n} \mu_i)^2$





Laplacian spectrum

$$\{0,\ 3-\sqrt{5},\ 2,\ 3,\ 3,\ 3+\sqrt{5}\}$$



NOT determined by the Laplacian spectrum are:

- number of triangles
- bipartite
- degree sequence
- girth

If G is regular of degree k, then $L = kI - A$
hence $\mu_i = k - \lambda_i$ for $i = 1 \dots n$

Properties determined by one spectrum are also determined by the other spectrum

For regular graphs the following are determined by the spectrum:

- number of vertices, edges, triangles; bipartite
- number of spanning trees, connected components

If G is regular of degree k, then $L = kI - A$
hence $\mu_i = k - \lambda_i$ for $i = 1 \ldots n$

Properties determined by one spectrum are also determined by the other spectrum

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- degree sequence
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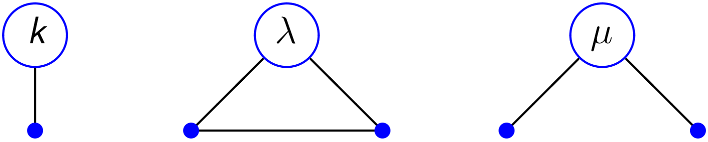
Strongly regular graph SRG(n, k, λ, μ)



$$A^2 = kI + \lambda A + \mu(J - I - A)$$

$$(A - rI)(A - sI) = \mu J, \quad r + s = \lambda - \mu, \quad rs = \mu - k$$

Every adjacency eigenvalue is equal to k, r, or s

Example SRG$(16, 9, 2, 4)$; Latin square graph

A	**C**	**B**	**D**
D	**A**	**C**	**B**
B	**D**	**A**	**C**
C	**B**	**D**	**A**

vertices: entries of the Latin square

adjacent: same row, column, or letter

adjacency spectrum $\{(-3)^6, 1^9, 9\}$

Example SRG$(16, 9, 2, 4)$; Latin square graph

A	C	B	D		A	C	B	D
D	A	C	B		C	A	D	B
B	D	A	C		B	D	A	C
C	B	D	A		D	B	C	A

vertices: entries of the Latin square

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adjacency spectrum $\{(-3)^6, 1^9, 9\}$

Theorem (Shrikhande, Bhagwandas 1965)

G is strongly regular

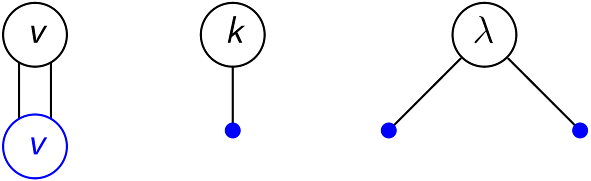
if and only if

G is regular and connected and has exactly three distinct eigenvalues, or G is regular and disconnected with exactly two distinct eigenvalues*

* i.e. G is the disjoint union of $m > 1$ complete graphs of order $\ell > 1$

Incidence graph of a symmetric (v, k, λ)-design

bipartite



Adjacency spectrum

$$\{-k, -\sqrt{k-\lambda}^{(v-1)}, \sqrt{k-\lambda}^{(v-1)}, k\}$$

Example Heawood graph, the incidence graph of the unique symmetric $(7, 3, 1)$-design (Fano plane)

$$A = \begin{bmatrix} O & N \\ N^\top & O \end{bmatrix} \quad \text{where} \quad N = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Spectrum $\{-3,\ -\sqrt{2}^{6},\ \sqrt{2}^{6},\ 3\}$

Theorem (Cvetković, Doob, Sachs 1984)

G is incidence graph of a symmetric (v, k, λ)-design if and only if G has adjacency spectrum

$$\{-k,\ -\sqrt{k-\lambda}^{(v-1)},\ \sqrt{k-\lambda}^{(v-1)},\ k\}$$

Corollary There exists a projective plane of order m if and only if there exists a graph with adjacency spectrum

$$\{-m-1,\ -\sqrt{m}^{m(m+1)},\ \sqrt{m}^{m(m+1)},\ m+1\}$$

For the following properties there exist a pair of cospectral regular graphs where one graph has the property and the other one not

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- being distance-regular of diameter $d \geq 3$ ($d \geq 4$ Hoffman 1963, $d = 3$ WHH 1992)

(A distance-regular graphs of diameter 2 is strongly regular)

For the following properties there exist a pair of cospectral regular graphs where one graph has the property and the other one not

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- having diameter $d \geq 2$ (WHH, Spence 1995)

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- being distance-regular of diameter $d \geq 3$ ($d \geq 4$ Hoffman 1963, $d = 3$ WHH 1992)
- having diameter $d \geq 2$ (WHH, Spence 1995)
- having a perfect matching ($\frac{n}{2}$ disjoint edges) (Blázsik, Cummings, WHH 2015)

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- having a perfect matching ($\frac{n}{2}$ disjoint edges) (Blázsik, Cummings, WHH 2015)
- having vertex connectivity ≥ 3 (WHH 2019)
- having edge connectivity ≥ 6 (WHH 2019)

For most NP-hard properties (chromatic number, clique number etc.) it is not hard to find a pair of cospectral regular graphs, where one has the property, and the other one not.

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Problem Does there exist a pair of cospectral regular graphs of degree k, where one has chromatic index (edge chromatic number) k, and the other $k+1$?

Characterizations from the spectral point of view

Proposition G has two distinct adjacency eigenvalues if and only if G is the disjoint union of complete graphs having the same order $m > 1$

Proposition G has two distinct Laplacian eigenvalues if and only if G is the disjoint union of complete graphs having the same order $m > 1$, possibly extended with some isolated vertices

Can we characterize the graphs with three distinct adjacency eigenvalues?

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If the graphs are regular and connected, then they are precisely the connected strongly regular graphs

If regularity is not assumed, then there exist other examples, but no characterization is known

Theorem (van Dam, WHH 1998)

A connected graph G has three distinct Laplacian eigenvalues if and only if μ and $\overline{\mu}$ are constant

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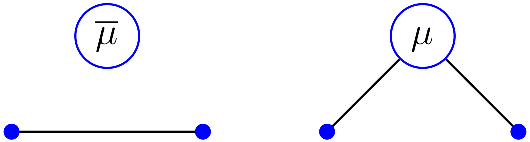

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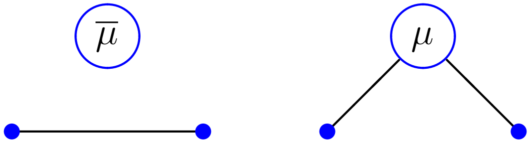
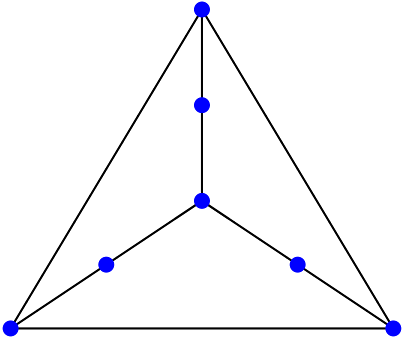


If G is regular of degree k, then $\overline{\mu} = n - 2k + \lambda$, and G is an SRG(n, k, λ, μ)

Example $n = 7,\ \mu = 1,\ \overline{\mu} = 2$



Laplacian spectrum $\{0,\ 3-\sqrt{2}^3,\ 3+\sqrt{2}^3\}$

Theorem (Cameron, Goethals, Seidel, Shult 1976)

A graph G has least adjacency eigenvalue ≥ -2 if and only if G is a generalized line graph, or G belongs to a finite set of exceptional graphs ($n \leq 36$)

Book: Spectral generalisations of line graphs,
Cvetković, Rowlinson, Simić 2004

Proposition G has least adjacency eigenvalue ≥ -1 if and only if G is the disjoint union of complete graphs

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Proof 1 $A + I$ is positive semi-definite, so it is the Gram matrix of a set of unit vectors with inner product 0 or 1

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Proof 1 $A + I$ is positive semi-definite, so it is the Gram matrix of a set of unit vectors with inner product 0 or 1

Proof 2 The path $P_3 =$  has spectrum $\{-\sqrt{2},\ 0,\ \sqrt{2}\}$, and by interlacing it can not be an induced subgraph of G

$$a^2 + b^2 = c^2$$

$a^2 + b^2 = c^2$ Pythagoras!

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$$e^{\pi i} + 1 = 0$$

$$a^2 + b^2 = c^2 \text{ Pythagoras!}$$

$$e^{\pi i} + 1 = 0 \text{ Euler!}$$

$$a^2 + b^2 = c^2 \quad \text{Pythagoras!}$$

$$e^{\pi i} + 1 = 0 \quad \text{Euler!}$$

$$1 = 0$$

$$a^2 + b^2 = c^2 \quad \text{Pythagoras!}$$

$$e^{\pi i} + 1 = 0 \quad \text{Euler!}$$

$$1 = 0\,!$$