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From Haemer's energy conjecture to Laplacian matrices

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The trace norm (or nuclear norm) of a matrix, defined as the sum of its singular values, plays a significant role in matrix analysis with applications ranging from graph theory to quantum mechanics. For Hermitian and real symmetric matrices, this norm coincides with the sum of absolute eigenvalues, often referred to as the "energy" in algebraic graph theory [1]. A fundamental problem in this domain concerns establishing lower bounds for the trace norm of special types of matrices with constrained entries.

This work originates from a conjecture by Haemers [2] regarding the minimal energy of Seidel matrices associated with simple graphs. Restated in matrix-theoretic terms:

Conjecture. Let A be an $n \times n$ real symmetric matrix with zero diagonal and ± 1 off-diagonal entries. Then $\|A\|_1 \geq 2n - 2$, with equality for $A = J_n - I_n$ (where J_n is the all-ones matrix and I_n the identity matrix).

This conjecture was resolved affirmatively in [3, 4]. We generalize this result to broader classes of matrices, establishing sharp lower bounds for the trace norm in terms of the entrywise L^1-norm. Our main theorems are:

Theorem 0.1 *Let $A = [a_{ij}]$ be a real symmetric $n \times n$ matrix with zero diagonal. Then*

$$\|A\|_1 \geq \frac{2}{n} \sum_{i,j} |a_{ij}|,$$

and this bound is tight for all n.

As an application, we obtain a new lower bound for the energy of signed graphs, generalizing known results for simple graphs:

Corollary. For every signed graph Σ, the energy satisfies

$$\mathcal{E}(\Sigma) \geq 2\bar{d},$$

where $\bar{d}$ is the average degree of Σ.

We combine Theorems 0.1 with standard duality arguments to establish tight upper bounds for the spectral norm distance of real symmetric matrices to diagonal matrices, which is the next theorem (where $\mathcal{D}_n$ denotes the space of real diagonal $n \times n$ matrices):

Theorem 0.2 *For any real symmetric matrix $A = [a_{ij}]$ of order $n \geq 2$,*

$$\min_{D \in \mathcal{D}_n} \|A - D\|_\infty \leq \frac{n}{2} \max_{i \neq j} |a_{ij}|,$$

with sharpness for all $n \geq 2$.

Notably, while Theorem 0.2 guarantees the existence of a diagonal matrix at spectral distance $\frac{n}{2}$ for matrices with entries in $[-1, 1]$, finding such a matrix numerically remains an open problem. But we fully characterize the equality cases in Theorem 0.2 and derive explicit formulas for the nearest diagonal matrices in these cases, revealing a surprising connection to graph Laplacians.

Detailed proofs and discussions appear in [5].

References

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[3] S. Akbari, M. Einollahzadeh, M. Karkhaneei, and M. Nematollahi, *Proof of a conjecture on the Seidel energy of graphs*, European Journal of Combinatorics, 103078, 2020.

[4] M. Einollahzadeh, and M. Nematollahi, *A short proof of Haemers' conjecture on the Seidel energy of graphs*, Linear Algebra and its Applications, 75–78, 2024.

[5] M. Einollahzadeh, *Minimum trace norm of real symmetric and Hermitian matrices with zero diagonal*, http://`arXivpreprintarXiv:2309.14958`