Large sets of t-designs

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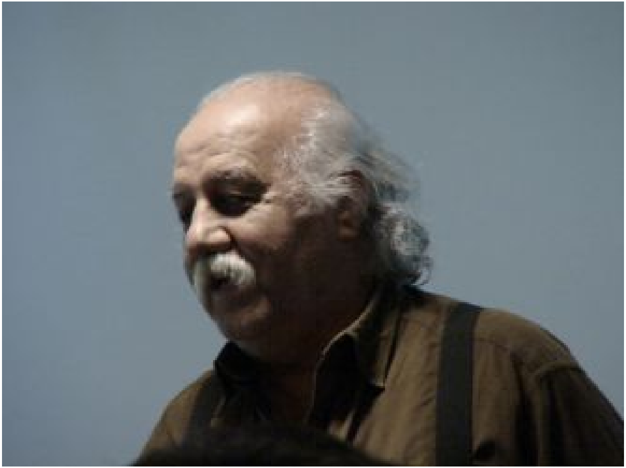
Isfahan, May 2014

Happy Birthdays!



Happy Birthdays!

t-(v, k, λ) designs

$0 \leq t \leq k \leq v$, $\lambda > 0$

X: A v-set

$P_k(X)$: k-subsets of X

A t-(v, k, λ) design is a pair

$$\mathcal{D} = (X, D)$$

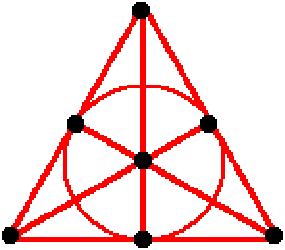
in which D is a subset of elements of $P_k(X)$ (called blocks) such that every t-subset of X appears in exactly λ blocks.

Example

$X = \{1, 2, 3, 4, 5, 6, 7\}$

$D = \{124, 235, 346, 457, 156, 267, 137\}$

(X, D) is a 2-$(7, 3, 1)$ design.



Example

$X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$D = \{123, 456, 789, 147, 258, 369, 168, 249, 357, 159, 267, 348\}$

(X, D) is a 2-$(9, 3, 1)$ design.

Large set

A large set of t-(v, k, λ) designs of size N,

$$\mathrm{L}S[N](t, k, v)$$

is a set of N disjoint t-(v, k, λ) designs (X, D_i) such that D_i partition $P_k(X)$.

Trivial necessary conditions:

$$N \Big| \binom{v-i}{k-i}, \qquad 0 \leq i \leq t.$$

Example

$X = \{1, 2, \ldots, 9\}$

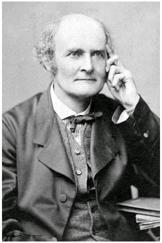
An $LS[7](2, 3, 9)$:

$$T_1 = \{124, 138, 157, 169, 237, 259, 268, 349, 356, 458, 467, 789\},$$
$$T_2 = \{129, 136, 145, 178, 235, 248, 267, 347, 389, 469, 568, 579\},$$
$$T_3 = \{123, 148, 159, 167, 249, 256, 278, 346, 358, 379, 457, 689\},$$
$$T_4 = \{126, 135, 147, 189, 234, 258, 279, 369, 378, 459, 468, 567\},$$
$$T_5 = \{128, 137, 149, 156, 239, 246, 257, 345, 368, 478, 589, 679\},$$
$$T_6 = \{125, 134, 168, 179, 238, 247, 269, 359, 367, 456, 489, 578\},$$
$$T_7 = \{127, 139, 146, 158, 236, 245, 289, 348, 357, 479, 569, 678\}.$$

Each (X, T_i) is a 2-(9,3,1) design.

Examples

- Cayley (1850): There is no LS[5](2, 3, 7).



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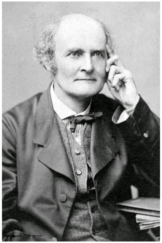

- Kramer and Mesner (1974): There exist exactly two nonisomorphic $\mathrm{LS}[7](2,3,9)$.

The existence

Trivial necessary conditions for the existence of

$LS[N](t,k,v)$ are

$$N \,\Big|\, \binom{v-i}{k-i}, \qquad 0 \le i \le t.$$

The known existence

- Baranyai (1975) and Hartman (1987): $t = 1$

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- Lu (1983) and Teirlinck (1991): $t = 2$ and $k = 3$
- Teirlinck (1987): For every t, there exists an $LS[N](t, t+1, v)$ for some v.

The necessary conditions

Notation: (m/n) is the remainder of division m by n

Theorem (Khosrovshahi and T, 2003)

$\mathrm{LS}[p^\alpha](t,k,v)$ is admissible if and only if there exist distinct positive integers ℓ_i $(1 \leq i \leq \alpha)$ such that $t \leq (v/p^{\ell_i}) < (k/p^{\ell_i})$.

Example

$LS[55](2,4,13)$ is admissible, Since

$$2 \leq (13/5) < (4/5)$$

and

$$2 \leq (13/11) < (4/11).$$

Partitionable sets

$\mathcal{B} \subseteq P_k(X)$ is called an

(N, t)-partitionable set

if it has a partition into N mutually t-balanced subsets.

$\mathrm{LS}[N](t, k, v) \Leftrightarrow P_k(X)$ is (N, t)-partitionable

Example

$\mathcal{B} = \{123, 124, 125, 136, 137, 145, 146, 157, 167, 234, 237, 248, 257, 258, 278, 346, 348, 368, 378, 456, 458, 567, 568, 678\}$

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$\mathcal{B}$ is a (3,2)-partitionable set.

The main lemma

Lemma (Ajoodani-Namini and Khosrovshahi)

Let $\mathcal{B}$ be (N, t_1)-partitionable and $\mathcal{C}$ be (N, t_2)-partitionable on disjoint sets. Then $\mathcal{B} * \mathcal{C}$ is $(N, t_1 + t_2 + 1)$-partitionable.

Example

$\mathcal{B} = \{1, 2, 3\}$

$\mathcal{B}_1 = \{1\}$, $\mathcal{B}_2 = \{2\}$, $\mathcal{B}_3 = \{3\}$

$\mathcal{B}$ is a (3,0)-partitionable set.

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$\mathcal{C} = \{45, 46, 47, 56, 57, 67\}$

$\mathcal{C}_1 = \{45, 67\}$, $\mathcal{C}_2 = \{46, 57\}$, $\mathcal{C}_3 = \{47, 56\}$

$\mathcal{C}$ is a (3,1)-partitionable set.

Example

$\mathcal{B} = \{1, 2, 3\}$

$\mathcal{B}_1 = \{1\}$, $\mathcal{B}_2 = \{2\}$, $\mathcal{B}_3 = \{3\}$

$\mathcal{B}$ is a (3,0)-partitionable set.

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$\mathcal{B} * \mathcal{C} = \{145, 245, 345, 146, 246, 346, 147, 247, 347, 156, 256, 356, 157, 257, 357, 167, 267, 367\}$

Example

Construction of an $\mathrm{LS}[2](2,3,10)$ from an $\mathrm{LS}[2](2,3,6)$:

$$\begin{aligned}
\mathcal{B}_1 &= P_3(\{1,\ldots,6\}),\\
\mathcal{B}_2 &= P_2(\{1,\ldots,5\}) * P_1(\{7,\ldots,10\}),\\
\mathcal{B}_3 &= P_1(\{1,\ldots,4\}) * P_2(\{6,\ldots,10\}),\\
\mathcal{B}_4 &= P_3(\{5,\ldots,10\}).
\end{aligned}$$

$\mathcal{B}_1, \mathcal{B}_4$: (2,2)-partitionable;

$\mathcal{B}_2, \mathcal{B}_3$: (2,2)-partitionable;

$\Rightarrow P_3(X)$: (2,2)-partitionable $\Rightarrow \mathrm{LS}[2](2,3,10)$

Recursive constructions

Theorem (T, 2005)

Let $X = \{1, 2, \ldots, v\}$ and also for $1 \le j \le v$, let $X_j = \{1, 2, \ldots, j\}$ and $Y_j = X \setminus X_j$. For $a \le i \le v - b - 1$, define

$$\mathcal{B}_i = P_a(X_i) * \{\{i+1\}\} * P_b(Y_{i+1}).$$

Then $\mathcal{B}_i$ provide a partitioning of $P_{a+b+1}(X)$.

A general Theorem

Theorem (Ajoodani-Namini, 1996)

If there exists an $\mathrm{LS}[p](t, k, v)$, then there exist $\mathrm{LS}[p](t+1, pk+i, pv+p+j)$ for all $0 \leq j < i \leq p-1$.

Halving

Theorem (Ajoodani-Namini, 1998)

All $LS[2](2, k, v)$ exist.

Theorem (Khosrovshahi and T, 2003)

All $LS[3](2, k, v)$ exist for $k < 81$.

Summary

N	t	k	v
$*$	1	$*$	$*$
$*$	2	3	$\neq 7$
2	2	$*$	$*$
2	≤ 5	≤ 16	$*$
2	6	$7, 8, 9$	$*$
3	2	≤ 80	$*$
3	≤ 4	≤ 8	$*$
5	2	≤ 5	$\neq 7$
5	3	≤ 5	$\neq 8$
7	2	≤ 6	$*$
11	2	≤ 10	$*$
29	2	≤ 5	$*$

Open problems

- Construct an LS[3]$(5, 6, 14)$. Five 5-$(14, 6, 3)$ designs are known, but the existence of LS[3]$(5, 6, 14)$ is unknown.

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- Construct LS[3]$(3^n - 2, 3^n - 1, 2.3^n - 2)$ for $n > 1$. If these large sets exist, then Khosrovshahi's conjecture on large sets of size 3 would follow.

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- Construct $LS[3](3^n - 2, 3^n - 1, 2.3^n - 2)$ for $n > 1$. If these large sets exist, then Khosrovshahi's conjecture on large sets of size 3 would follow.
- Are there general theorems similar to Ajoodani's theorem for large sets of prime power sizes?

Over finite fields

Cameron (1974) and Delsarte (1976)

A t-$(v, k, \lambda; q)$ design is a set $\mathcal{B}$ of k-dimensional subspaces of an v-dimensional vector space V over the finite field $\mathbb{F}_q$ with q elements such that each t-dimensional subspace of V is contained in exactly λ members of $\mathcal{B}$.

Large sets

Ray-Chaudhuri and Schram (1994)

An $\mathrm{LS}_q[N](t, k, v)$ large set is a set of N disjoint t-$(v, k, \lambda; q)$ designs such that their union forms the complete set of all k-dimensional subspaces of $V = \mathbb{F}_q^n$.

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Braun, Kohnert, ostergard, Wassermann (2014)

$LS_2[3](2, 3, 8)$ exists.