
The First IPM-Isfahan Workshop on Combinatorics
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Generalized Hadamard matrices and applications

H. Kharaghani

University of Lethbridge
Canada

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University of Isfahan

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- Some application of GH matrices

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Example: A $(3, 6; 2)$-difference matrix over $(\mathbb{Z}_3, +)$

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 2 & 1 & 0 \\ 0 & 2 & 2 & 0 & 1 & 1 \\ 2 & 2 & 0 & 1 & 1 & 0 \\ 2 & 0 & 2 & 1 & 0 & 1 \end{pmatrix}$$

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$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 4 & 1 & 3 \\ 0 & 3 & 1 & 4 & 2 \\ 0 & 4 & 3 & 2 & 1 \end{pmatrix}$$

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A $(5,5;1)$-difference matrix over $(\mathbb{Z}_5,x)$

$$B = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 \\ 1 & \omega^2 & \omega^4 & \omega & \omega^3 \\ 1 & \omega^3 & \omega & \omega^4 & \omega^2 \\ 1 & \omega^4 & \omega^3 & \omega^2 & \omega \end{pmatrix}$$

A $(15, 5; 1)$-difference matrix over $(\mathbb{Z}_{15}, +)$:

$$\text{Take } B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 5 & 7 & 9 & 12 & 4 & 1 \\ 6 & 3 & 14 & 10 & 7 & 13 & 4 \\ 10 & 6 & 1 & 11 & 2 & 7 & 12 \end{pmatrix},$$

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$$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 3 & 1 \\ 0 & 3 & 1 & 2 \end{pmatrix},$$

where $2+3=1$.

A necessary condition for the existence:

A $(g, k; \lambda)$-difference matrix does not exist if $k > \lambda g$.

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Conjecture: There is no $(q, q; 1)$-difference matrix for any non-prime power q.

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Auxiliary matrices corresponding to difference matrices:

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If there is a $(g, k; 1)$-difference matrix, then there is a set of $k-1$ Mutually Suitable Latin Squares of size g.

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Example: A $\mathrm{GH}(3,2)$ generalized Hadamard matrix over the cyclic group $\mathbb{Z}_3$

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \omega & \omega^2 & 1 & \omega^2 & 1 & \omega \\ \omega & 1 & \omega^2 & \omega^2 & \omega & 1 \\ 1 & \omega^2 & \omega^2 & 1 & \omega & \omega \\ \omega^2 & \omega^2 & 1 & \omega & \omega & 1 \\ \omega^2 & 1 & \omega^2 & \omega & 1 & \omega \end{pmatrix}$$

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$c = ab$.

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The multiplication table for the finite field $\mathbf{GF}(q)$ is a $(q, q; 1)$-difference matrix and so a generalized Hadamard matrix over EA(q).

For any Group G of prime power order q and any integer $t > 0$, there is a GH(q, q^{2t-1}).

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Consider the normalized $\mathrm{GH}(4,1) = \begin{pmatrix} I & I & I & I \\ I & c & a & b \\ I & b & c & a \\ I & a & b & c \end{pmatrix}$, $c = ab$, over $\mathbb{Z}_2 \times \mathbb{Z}_2$.

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$$C = \begin{pmatrix} I & b & c & a \\ I & b & -c & -a \\ I & -b & c & -a \\ I & -b & -c & a \end{pmatrix} \quad D = \begin{pmatrix} I & a & b & c \\ I & a & -b & -c \\ I & -a & b & -c \\ 1 & -a & -b & c \end{pmatrix}$$

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Note $I + a + b + c = J$, J is the 4×4 matrix of all ones.

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COMBINATORICS
NAUTY

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- Any $GH(q, 1)$ over an $EA(q)$ leads to some very interesting objects.

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Let p be a prime number. Then there is a $p+1$-class symmetric association scheme of order p^2 which is often collapsable to smaller classes.

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The Generalized Bush-type of order 16 is:

$$H = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 0 & 2 & 3 & 1 & 0 & 3 & 1 & 2 \\
0 & 0 & 0 & 0 & 1 & 0 & 3 & 2 & 2 & 0 & 1 & 3 & 3 & 0 & 2 & 1 \\
0 & 0 & 0 & 0 & 2 & 3 & 0 & 1 & 3 & 1 & 0 & 2 & 1 & 2 & 0 & 3 \\
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2 & 3 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 3 & 3 & 1 & 0 & 2 \\
3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 3 & 0 & 1 & 3 & 2 & 0 \\
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3 & 1 & 0 & 2 & 1 & 2 & 0 & 3 & 0 & 0 & 0 & 0 & 2 & 3 & 0 & 1 \\
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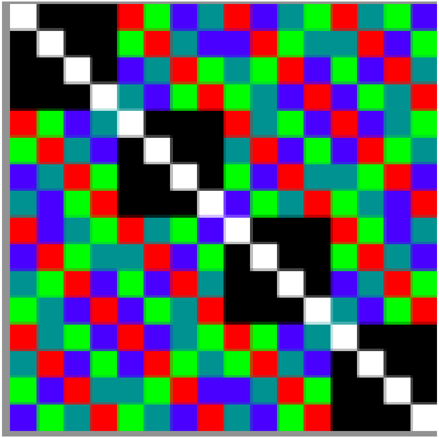

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Main result: For any prime number p there is a $(p+1)$-class symmetric association scheme collapsible to smaller class symmetric association scheme of order p^2.

A BIG **Thank you** to the organizers

Happy Birthday Reza!