Gluing derived equivalences together with bimodules

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The University of Isfahan, 2014-09-18 (120 min)

Throughout

I : a small cat

$\Bbbk$: a comm ring (a field when der eq is dealt with)

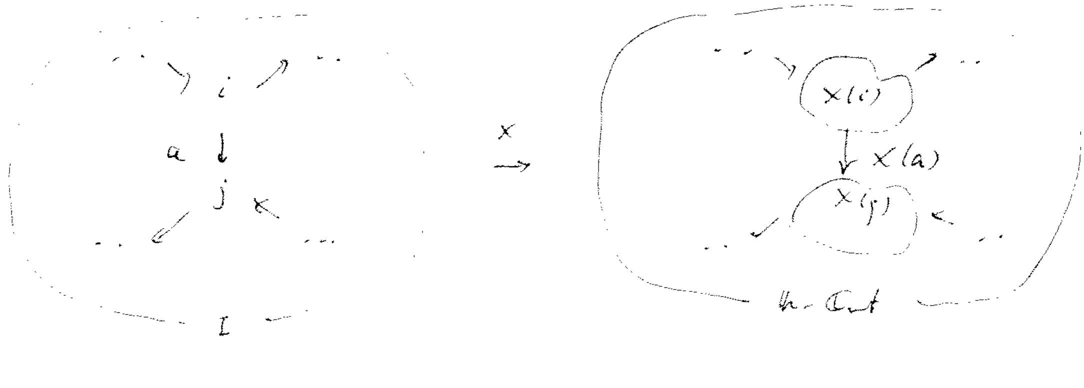
$\Bbbk$-Mod := the cat of $\Bbbk$-modules

$\Bbbk$-Cat := the 2-cat of small $\Bbbk$-cat^s, $\Bbbk$-fun^s and nat. tr^s.

1. Grothendieck constructions Gr
2. Gluing derived equivalences together with functors
3. Generalization of Gr
4. Gluing derived eq^s together with bimod^s
5. Examples

1. Grothendieck constructions

$X : I \to \Bbbk\text{-Cat}$ a (colax) functor.

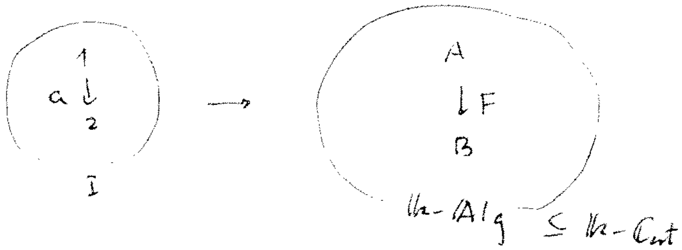


Exm 1 $I = G$: a gp = a cat with a single obj $*$ with all morph^s invertible

$$G \xrightarrow{X} k\text{-}\mathbb{C}\mathrm{at}$$

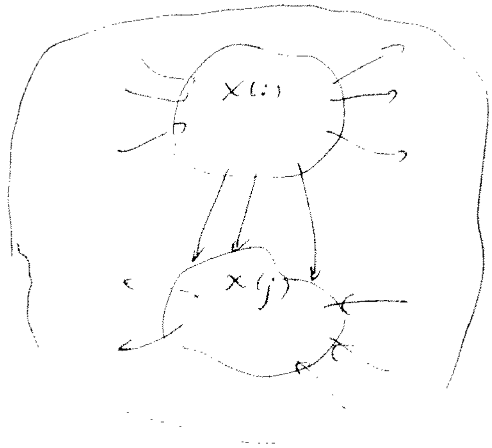
$$a \circlearrowleft * \longmapsto \mathcal{C} \circlearrowleft X(a) \quad = \quad \text{a } G\text{-action on a } k\text{-cat } \mathcal{C}\ (= X(*))$$

Exm 2 $I = \mathbb{P}(1 \xrightarrow{a} 2)$ the free cat of the quiver $1 \xrightarrow{a} 2$



Then a k-cat $Gr(X)$ is constructed ← the Grothendieck construction of X

$Gr(X)$:



For **Exm 1** $Gr(X) = \mathcal{C}/G$, the orbit cat of $\mathcal{C}$ by G

For **Exm 2** $Gr(X) = \begin{bmatrix} A & 0 \\ \tilde{F} & B \end{bmatrix}$, where

$\tilde{F} := B_F = {}_B B(F(-), -)_A$: a B-A-bimod

a very special bimodule

Prob 1 For ${}_B M_A$ bimodule, $\exists X$: something st.

$$\left\{ \exists\, Gr(X) \simeq \begin{bmatrix} A & 0 \\ M & B \end{bmatrix} \ ? \quad \text{(generalized version of } Gr) \right.$$

generalized

Prob 1′ For k-species $\mathcal{S}$ over a quiver Q, i.e.,

$$\mathcal{S} = (\mathcal{S}(i), \mathcal{S}(a))_{\substack{i \in Q_0 \\ a \in Q_1}} \quad \begin{cases} \mathcal{S}(i) : k\text{-alg} \\ \mathcal{S}(a) : \mathcal{S}(j)\text{-}\mathcal{S}(i)\text{-bimodule for } i \xrightarrow{a} j \end{cases}$$

$\exists X$: something st.

$\exists\, Gr(X) \cong T(\mathcal{S})$: the tensor alg of $\mathcal{S}$?

(gen^{d} ver of Gr)

$$:= \underbrace{\left(\prod_{i \in Q_0} \mathcal{S}(i)\right)}_{R} \oplus \underbrace{\left(\bigoplus_{a \in Q_1} \mathcal{S}(a)\right)}_{V} \oplus V^{\otimes_R 2} \oplus V^{\otimes_R 3} \oplus \cdots$$

2. Gluing der eq together with functors

Until now :

Q $X, X' : I \to k\text{-}\mathbb{C}at$ (colax) fun^s.

When $X(i) \overset{der}{\sim} X'(i)$ $(i \in I_0)$ $\Rightarrow$ $Gr(X) \overset{der}{\sim} Gr(X')$?

"gluing"

Ans (Thm) "$X \overset{der}{\sim} X'$" $\dashrightarrow$ $Gr(X) \overset{der}{\sim} Gr(X')$

$\Updownarrow$ $\Uparrow$

$X' \simeq \exists\, \mathcal{T} \xhookrightarrow{I\text{-equivar}} K^b(\mathrm{prj}\, X)$

tilting

$\mathcal{T}(i) \hookrightarrow K^b(\mathrm{prj}\, X(i))$ $(i \in I_0)$
tilting

In <u>Exm 1</u> $\quad X' : G \to k\text{-Cat}$. Then
$\qquad\qquad * \mapsto \mathcal{C}'$

$$X \overset{\mathrm{der}}{\sim} X' \qquad\qquad \mathcal{C}/G \overset{\mathrm{der}}{\sim} \mathcal{C}'/G$$

$$\Updownarrow \qquad\qquad\qquad \Uparrow$$

$$\boxed{\begin{tikzcd}
\mathcal{C}' \arrow[r, "\phi", "\sim"'] \arrow[d, "X'(a)"'] \arrow[dr, phantom, "\swarrow \psi(a)"] & \overset{\exists\ \text{tilting}}{T} \arrow[r, hook] \arrow[d, "T(a)"] \arrow[dr, phantom, "\swarrow \rho(a)\ \sim"] & K^b(\mathrm{prj}\,\mathcal{C}) \arrow[d, "K^b(\mathrm{prj}\,X(a))"] \\
\mathcal{C}' \arrow[r, "\phi", "\sim"'] & T \arrow[r, hook] & K^b(\mathrm{prj}\,\mathcal{C})
\end{tikzcd} \quad (a \in G).}$$

In <u>Exm 2</u> $\quad X' : \begin{tikzcd} 1 \arrow[d, "a"'] \\ 2 \end{tikzcd} \mapsto \begin{tikzcd} A' \arrow[d, "F'"] \\ B' \end{tikzcd}$. Then

$$X \overset{\mathrm{der}}{\sim} X' \qquad\qquad \begin{bmatrix} A & 0 \\ \tilde{F} & B \end{bmatrix} \overset{\mathrm{der}}{\sim} \begin{bmatrix} A' & 0 \\ \tilde{F}' & B' \end{bmatrix}$$

$$\Updownarrow \qquad\qquad\qquad \Uparrow$$

$$\boxed{\begin{tikzcd}
A' \arrow[r, "\sim"] \arrow[d, "F'"'] \arrow[dr, phantom, "\swarrow\sim"] & \overset{\text{tilting}}{T_1} \arrow[r, hook] \arrow[d, "T(a)"] \arrow[dr, phantom, "\swarrow\sim"] & K^b(\mathrm{prj}\,A) \arrow[d, "K^b(\mathrm{prj}\,F)"] \\
B' \arrow[r, "\sim"] & \underset{\text{tilting}}{T_2} \arrow[r, hook] & K^b(\mathrm{prj}\,B)
\end{tikzcd}} \qquad \begin{matrix} \leftrightarrow & A \overset{\mathrm{der}}{\sim} A' \\ \\ \leftrightarrow & B \overset{\mathrm{der}}{\sim} B' \end{matrix}$$

<u>Prob 2</u> When $A \overset{\mathrm{der}}{\sim} A'$, $B \overset{\mathrm{der}}{\sim} B'$, ${}_B M_A$, ${}_{B'}M'_{A'}$: bimodules

$$\underset{\text{"gluing"}}{\Longrightarrow} \quad \begin{bmatrix} A & 0 \\ M & B \end{bmatrix} \overset{\mathrm{der}}{\sim} \begin{bmatrix} A' & 0 \\ M' & B' \end{bmatrix} \ ?$$

Generalize the <u>Thm</u> above to include this case.

$\downarrow$ generalized

<u>Prob 2'</u> The corresponding problem for k-species $\mathcal{S}$ over a quiver Q and the tensor cat $T(\mathcal{S})$ of $\mathcal{S}$.

3. Generalization of Gr

Regard $\begin{bmatrix} A & 0 \\ M & B \end{bmatrix} = \begin{bmatrix} X(1) & 0 \\ X(a) & X(2) \end{bmatrix}$ for $X : \begin{array}{c} 1 \\ a \downarrow \\ 2 \end{array} \mapsto \begin{array}{c} X(1) \\ \downarrow X(a) \\ X(2) \end{array}$,

where ${}_{X(2)}X(a)_{X(1)}$ is a bimodule.

This leads us to consider the following.

<u>Dfn</u> $\Bbbk\text{-}\mathbb{C}\mathrm{at}^b$ is a bicategory defined as follows.

<u>obj</u> small $\Bbbk$-cat's.

<u>1-mor</u> $\forall A, B \in (\Bbbk\text{-}\mathbb{C}\mathrm{at}^b)_0$, 1-mor $A \to B$ are the B-A-bimod^s

${}_B M_A : A^{op} \times B \to \Bbbk\text{-Mod}$ ($\Bbbk$-bilinear functors)

$$\begin{array}{ccc} (i', j) & & M(i', j) \\ (a,b) \downarrow & \mapsto & \downarrow M(a,b) \\ (i, j') & & M(i, j') \end{array}, \qquad \mathbb{1}_A = {}_A A(-,-)_A$$

<u>2-mor</u> $\forall A, B \in (\Bbbk\text{-}\mathbb{C}\mathrm{at}^b)_0$, $\forall {}_B M_A, {}_B M'_A : A \to B$ 1-mor^s

2-mor^s $M \Rightarrow M'$ are the bimodule morphisms.

<u>Rmk</u> Let ${}_D N_C, {}_C M_B, {}_B L_A$: bimodules. Then

$(N \otimes_C M) \otimes_B L \underset{\neq}{\xrightarrow{\sim}} N \otimes_C (M \otimes_B L)$ --- <u>weak</u> associativity (up to nat iso)

$M \otimes_B B \underset{\neq}{\xrightarrow{\sim}} M$, $C \otimes_C M \underset{\neq}{\xrightarrow{\sim}} M$ --- <u>weak</u> unitality

$\Downarrow$

<u>bicat</u>, not 2-cat

Ans to Prob 1, 1'

Let X : a lax functor : $I \to \underset{\sim}{k\text{-Cat}^b}$

For Prob 1 $\quad X : \underset{2}{\downarrow a} \mapsto \begin{array}{c} A \\ \downarrow {}_BM_A \\ B \end{array}$

For Prob 1' $\quad X :$
$$\begin{tikzcd} I \arrow[r] & k\text{-Cat}^b \\ \Omega \arrow[u, hook] \arrow[ur, "\delta"'] \arrow[ur, phantom, bend left, "\circlearrowright"] \end{tikzcd}$$

Dfn $\quad Gr(X)$ is a k-cat defined as follows.

obj $\quad \boxed{Gr(X)_0 := \coprod_{i \in I_0} X(i)_0} = \{ {}_ix = (i,x) \mid i \in I_0,\ x \in X(i)_0 \}$

mor $\quad \forall\, {}_ix, {}_jy \in Gr(X)_0$, $\boxed{Gr(X)({}_ix, {}_jy) := \bigoplus_{a \in I(i,j)} X(a)(x,y)}$

comp $\quad \forall\, {}_ix \xrightarrow{f} {}_jy \xrightarrow{g} {}_kz$

$$\begin{tikzcd} Gr(X)({}_jy, {}_kz) \times Gr({}_ix, {}_jy) \arrow[rr] & & Gr({}_ix, {}_kz) \\ X(b)(y,z) \times X(a)(x,y) \arrow[u, hook] \arrow[rr] \arrow[d, "\otimes"'] \arrow[urr, phantom, "\circlearrowleft"] & & X(ba)(x,z) \arrow[u, hook] \\ X(b) \otimes_{X(j)} X(a)\,(x,z) \arrow[urr, "\theta_{b,a}"'] \arrow[ur, phantom, "\circlearrowright"] & & \end{tikzcd}$$

$(g_b, f_a) \in X(b)(y,z) \times X(a)(x,y)$, $\quad X(ba)(x,z) \ni \theta(g_b \otimes f_a)$

$\theta_{b,a}$ (str of lax fun X)

$(g_b, f_a) \mapsto g_b \otimes f_a \in X(b) \otimes_{X(j)} X(a)\,(x,z)$

Thus

$$\boxed{gf := \Big(\sum_{\substack{a \in I(i,j) \\ b \in I(j,k) \\ c = ba}} \theta(g_b \otimes f_a) \Big)_{c \in I(i,k)}}.$$

Rmk. This is a generalization of older version for

colax fun $Y : I \to$ lk-Cat.

$$Y \longmapsto \tilde{Y} \qquad \tilde{Y}(a) := \underset{Y(j)}{Y(j)}(\underbrace{Y(a)(-), -}_{\uparrow})_{Y(i)} \quad \text{bimod.} \; (i \xrightarrow{a} j)$$

colax fun   lax fun   position of contravariant $\tilde{Y} : I \to$ lk-Cat^b

Then $Gr(Y) \cong Gr(\tilde{Y})$.
old   new

4. Gluing der eq together with bimodules

Dfn. Let $X, X' : I \to$ lk-Cat^b lax functors. Then

$X \overset{\text{der}}{\sim} X'$

$:\Leftrightarrow \mathcal{D}(\mathrm{Mod}\, X) \simeq \mathcal{D}(\mathrm{Mod}\, X')$

in a 2-cat Lax$(I$, lk-Tri$^b)$ of lax functors

$I \to$ lk-Tri^b, where lk-Tri^b is a bicat of small

triangulated lk-cat's, similarly defined as lk-Cat^b, and

$\mathcal{D}(\mathrm{Mod}\, X) := \mathcal{D}(\mathrm{Mod}) \circ X$:

$$I \xrightarrow{X} \text{lk-Cat}^b \xrightarrow{\mathcal{D}(\mathrm{Mod})} \text{lk-Tri}^b$$

$$\boxed{{}_{\mathcal{C}'}M_{\mathcal{C}} \quad \begin{array}{c}\mathcal{C}\\ \downarrow \\ \mathcal{C}'\end{array}} \longmapsto \boxed{\begin{array}{l}\mathcal{D}(\mathrm{Mod}\,\mathcal{C}) \\ \downarrow \; \mathcal{D}(\mathrm{Mod}\,M) := \mathcal{D}(\mathrm{Mod}\,\mathcal{C})(-, - \overset{\mathbb{L}}{\underset{\mathcal{C}'}{\otimes}} M) \\ \mathcal{D}(\mathrm{Mod}\,\mathcal{C}')\end{array}}$$

Thm. $X, X' \in \mathrm{Lax}(I, \Bbbk\mathrm{Cat}^b)_0$. Then

$$\begin{tikzcd}
X \overset{\text{der}}{\sim} X' \arrow[d, Rightarrow] & \mathrm{Gr}(X) \overset{\text{der}}{\sim} \mathrm{Gr}(X') \\
\boxed{X' \simeq \exists T \xhookrightarrow{\exists I\text{-equiv}} K^b(\mathrm{prj}\, X) \quad T:\ \text{tilting lax subfunctor}} \arrow[ur, dashed, Rightarrow, "\text{if } X:\ \text{locally bounded}"'] &
\end{tikzcd}$$

(this holds for $I = PQ$, Q: loc finite quiver)

(i.e. $\{a \in I(i,j) \mid X(a)(x,y) \neq 0\}$: finite $\forall i, j \in I_0$, $x \in X(i)_0$, $y \in X(j)_0$)

<u>$K^b(\mathrm{prj}\, X)$</u>

$$\begin{array}{ccc} i & & K^b(\mathrm{prj}\, X(i)) \\ a \downarrow & \longmapsto & \Big|\ \mathcal{D}(\mathrm{Mod}\, X(a))\Big|_{K^b(\mathrm{prj}\, X(i))^{op} \times K^b(\mathrm{prj}\, X(j))} \\ j & & K^b(\mathrm{prj}\, X(j)) \end{array}$$

Def. T: a <u>tilting lax subfunctor</u> for X

(Note: $V \in K^b(\mathrm{prj}\, X(j))$ $\rightsquigarrow$ $V \overset{\mathbb{L}}{\otimes}_{T(j)} T(a) = V \otimes_{T(j)} T(a)$)

$:\Leftrightarrow$
- $\forall i \in I_0$, $T(i) \hookrightarrow K^b(\mathrm{prj}\, X(i))$
- $\forall i, j \in I_0$, $\forall U \in T(i)_0$, $\forall V \in T(j)_0$, $\forall a \in I(i,j)$, $\forall n \neq 0$,
$$\mathcal{D}(\mathrm{Mod}\, X(i))(U, V \overset{\mathbb{L}}{\otimes}_{T(j)} T(a)[n]) = 0$$
- $\forall i \in I_0$, $\mathrm{thick}\, T(i) = K^b(\mathrm{prj}\, X(i))$ (i.e. $\forall x \in X(i)_0$ $X(i)(-, x) \in \mathrm{thick}\, T(i)$)

Cor 1 (Ans to Prob 2) A_1, A_2: $\Bbbk$-alg's ${}_{A_2}M_{A_1}$: a bimod

$T_i \in K^b(\mathrm{prj}\, A_i)$: a tilting cpx $(i = 1, 2)$.

Assume $\mathcal{D}(\mathrm{Mod}\, A_1)(T_1, T_2 \overset{\mathbb{L}}{\otimes}_{A_2} M[n]) = 0$ $\forall n \neq 0$

Then

$$\begin{bmatrix} A_1 & 0 \\ M & A_2 \end{bmatrix} \overset{\text{der}}{\sim} \begin{bmatrix} \mathrm{End}_{K^b(\mathrm{prj}\, A_1)}(T_1) & 0 \\ \mathcal{D}(\mathrm{Mod}\, A_1)(T_1, T_2 \overset{\mathbb{L}}{\otimes}_{A_2} M) & \mathrm{End}_{K^b(\mathrm{prj}\, A_2)}(T_2) \end{bmatrix}$$

Cor 1' (Ans to Prob 2')

Q : a quiver with bipartite orientation

$I = \mathbb{P}Q$, S, S' : $\Bbbk$-species over Q

$X := X_S$, $X' := X_{S'} \in \mathrm{Lax}(I, \Bbbk\text{-}\mathbb{C}\mathrm{at}^b)$

Assume that

$\forall i \in Q_0$, $\exists T_i$: a tilting cpx of $K^b(\mathrm{prj}\, X(i))$

$\exists F(i) : X'(i) \xrightarrow{\sim} \mathrm{End}_{K^b(\mathrm{prj}\, X(i))} T_i$ an iso of $\Bbbk$-alg^s

s.t. $\forall i, j \in Q_0$, $\forall a \in Q(i,j)$

$$\mathcal{D}(\mathrm{Mod}\, X(i))(T_i, T_j \overset{\mathbb{L}}{\otimes}_{X(j)} X(a)[n]) \cong \begin{cases} {}_{X'(j)}X'(a)_{X'(i)} & (n=0) \\ 0 & (n \neq 0) \end{cases}$$

$\uparrow$ $X'(j)$-$X'(i)$-bimod through $F(i), F(j)$.

Then

$$T(S) \overset{\text{der}}{\sim} T(S')$$

Outline of Proof.

$$\begin{tikzcd} \mathrm{Lax}(I, \Bbbk\text{-}\mathbb{C}\mathrm{at}^b) \arrow[r, bend left, "Gr"] \arrow[r, phantom, "\perp"] & \Bbbk\text{-}\mathbb{C}\mathrm{at} \arrow[l, bend left, "\Delta"] \end{tikzcd} \qquad \left(\begin{array}{c} i \xrightarrow{a} j \text{ in } I \\ I\Delta(\mathcal{C}) \\ \mathcal{C} \xrightarrow[\mathcal{C}(-)]{} \mathcal{C} \end{array} \right)$$

unit $\zeta : \mathbb{1} \Rightarrow \Delta \circ Gr$

$(\zeta_X)_{X \in \mathrm{Lax}}$ $\quad \zeta_X : X \longrightarrow \Delta(Gr(X))$ turns out to be "I-covering"

$$(P_X, \phi_X)_{i,j}(x,y) : \bigoplus_{a \in I(i,j)} X(a)(x,y) \xrightarrow[\sim]{id} Gr(X)(P(i)x, P(j)y)$$

$$\begin{tikzcd} T(i) \arrow[d, phantom, "\in"] \\ K^b(\mathrm{prj}\, X(i)) \arrow[d, "- \overset{\mathbb{L}}{\otimes}_{X(i)} \overline{P(i)}"] \\ K^b(\mathrm{prj}\, Gr(X)) \end{tikzcd}$$

This induces an I-precovering $\qquad \overline{P(i)} := Gr(X)(-, P(i)(-))$

$$\mathcal{D}(\mathrm{Mod}\, X) \to \Delta(\mathcal{D}(\mathrm{Mod}\, Gr(X)))(- \overset{\mathbb{L}}{\otimes}_{X(i)} \overline{P(i)}, - \overset{\mathbb{L}}{\otimes}_{X(j)} \overline{P(j)})$$

Then

$$\mathcal{T} := \left\langle \bigcup_{i \in I_0} T(i) \overset{\mathbb{L}}{\otimes}_{X(i)} \overline{P(i)} \right\rangle \subseteq K^b(\mathrm{prj}\, Gr(X))$$ is a required tilting subcat for $Gr(X)$

iso closure

that gives $Gr(X) \overset{\text{der}}{\sim} Gr(X')$.

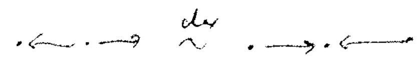
5. Exm

Presentation of bimodules

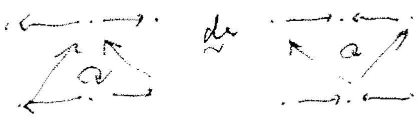
$$\begin{tikzcd} A: & 1 \arrow[r, "a"] \arrow[d, "m"'] \arrow[dr, phantom, "\circlearrowleft"] & 2 \arrow[d, "n"] & 3 \arrow[l, "b"'] \\ B: & 1' \arrow[r, "a'"'] & 2' \arrow[r, "b'"'] & 3' \end{tikzcd}$$

${}_B M_A$: the bimodule gen by m and n with relation $a'm = na$

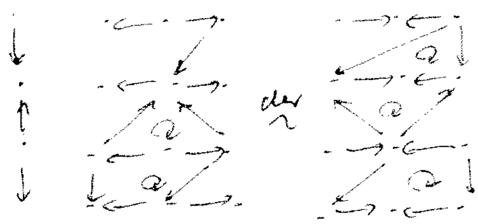
M = k-vector sp with basis m, $(a'm)$ $(b'a'm)$
n, na, nb, $b'n$, $b'na$, $b'nb$

(1) 

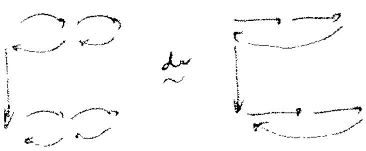
(a) 

(b) 

(c) 

(d) I 

(2) Brauer tree alg's

(a) 

(b) I