# Basic Algebraic Structures

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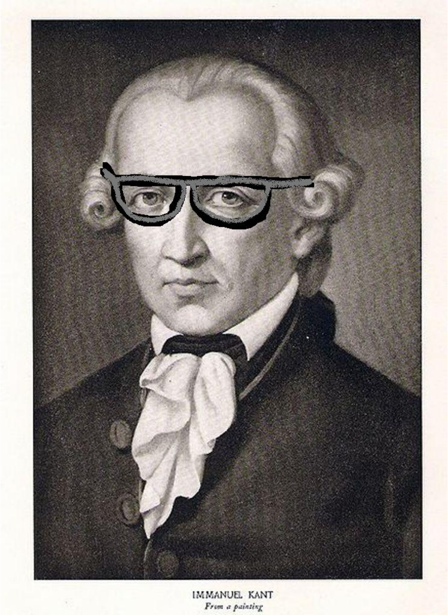



Glasses through which we view mathematics.

# Groups

Formulas for equations of degrees 2,3,4

$$x^2 - ax + b = 0; \qquad x_{1,2} = \frac{a \pm \sqrt{a^2 - 4b}}{2}$$

Cubic equation: Cardano

Quartic equation: Ferrari

Equation of degree 5?

Fantastic observation of J.-L. Lagrange (1736 - 1813):
key to these formulas is symmetry of roots.

# Groups

Theory of symmetries : Group Theory

$\alpha, \beta$ symmetries

Perform $\alpha$ and then perform $\beta \rightsquigarrow$ multiplication $\alpha\beta$

Undo $\alpha \rightsquigarrow \alpha^{-1}$

There is a very special symmetry: Id, nothing is moved

# Groups

A. - L. Cauchy, P. Ruffini (almost did)

N. H. Abel (1802 - 1829) : did!

No formula for equations of degree $\geq 5$.

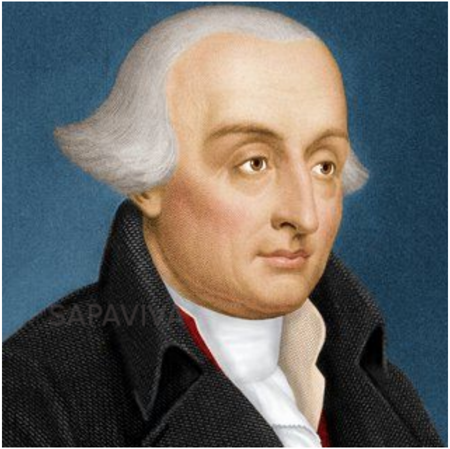
Evariste Galois (1811 - 1832)

Complete understanding of how symmetries (Galois Groups) control equations.

The right question about any object:
What is the group of symmetries?

Hermann Weyl: the most influential paper in the history of mathematics.

Evariste Galois



Joseph-Louis Lagrange

# Groups

20th century, Elementary Particles

"If God exists then he knows Group Theory"

$G$ finite group; $H \lhd G$ normal subgroup

$G$ is built from $H$, $G/H$

"Bricks": simple groups

The Biggest Program in Algebra (started in 1837, lasted for $>$ 150 years): find all finite simple groups.

# Groups

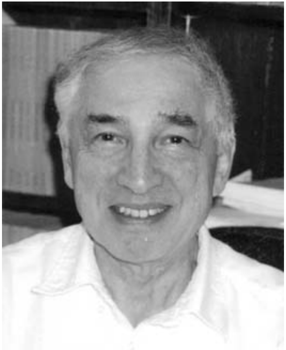
E. Galois (1837) : $A_n, n \geq 5$

Jordan, Dixon (1870) : groups of matrices

Mathieu (1860) : 5 "exceptional" groups

More and more to come

W. Feit - J. Thompson (1960) : all simple groups have even number of elements




## R. Brauer Program

$g \in G, g^2 = 1$

$C(g) = \{x \in G | xg = gx\}$ centralizer.

smaller than $G$, hence "known" $\stackrel{?}{\Rightarrow} G$.

Griess-Fischer (1980): the biggest sporadic simple group, known as THE MONSTER

Huge: representable by matrices of order 196883.

$J(q) = q^{-1} + 0 + 196884q + \cdots$
normalized modular series.

Moonshine Conjecture (McKay, Norton, Thompson)

There exists a representation of the Monster

$$V = V_{-1} + V_0 + V_1 + V_2 + \cdots$$

such that

1) $\sum(\dim V_i)q^i = J(q)$
2) for any $g \in G$, $\sum_i \mathrm{Tr}_{V_i}(g)q^i$ is modular .

# Groups

1982 - The Classification announced to be finished.

Actually completely published in 2011.

More than 10,000 pages.

## Infinite Groups

M. Dehn (1906) $\quad G = \langle a_1, \ldots, a_m | r_1 = 1, \ldots, r_s = 1 \rangle$
$v(a_1, \ldots, a_m) \overset{?}{=} w(a_1, \ldots, a_m)$.

### Algorithmic Problems

Definition of an algorithm, Turing machines.

# Groups

P.S. Novikov (1952): groups with undecidable word problem.

W. Burnside (1902): $G = \langle a_1, \ldots, a_m \rangle$, there exists $n \geq 1$ such that $g^n = 1$ for all $g \in G \overset{?}{\Rightarrow} |G| < \infty$.

What makes a group finite?

Big impact on Infinite Algebra and on my life.

# Groups

Infinite Groups

Hopelessly Infinite

Residually Finite

Examples of Novikov-Adian, Ol'shansky

$\downarrow$

Geometric Group Theory

Links to Number Theory,
Graph Theory
Restricted Burnside Problem

$G = \langle a_1, \ldots, a_m \rangle \to$ Cayley Graph

$\Gamma = (V, E)$

$V = G$

$g_2$

$g_1$

$g_2 = a_i^{\pm 1} g_1$

# Groups

A connected graph is a metric space.

$G$ acts on $\Gamma$ by isometries.

Gromov (80s): Group as a Geometric Object.

## Rings

$$(x^2 + y^2)(z^2 + t^2) = (xz - yt)^2 + (xt + yz)^2$$

Complex numbers $\mathbb{C} = \mathbb{R} + \mathbb{R}i, i^2 = -1.$

$$(x + yi)(z + ti) = (xz - yt) + (xt + yz)i$$

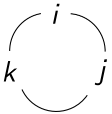
What about a product of $x^2 + y^2 + z^2$'s?

# Quaternions

W.R. Hamilton (1805-1865) October 16, 1843 walking from the Observatory to the University:

$$\mathbb{H} = \mathbb{R}\cdot 1 + \mathbb{R}\cdot i + \mathbb{R}\cdot j + \mathbb{R}\cdot k$$

$$i^2 = j^2 = k^2 = -1$$
$$ij = k, \quad jk = i, \quad ki = j$$
$$ij = -ji, \quad ik = -ki, \quad jk = -kj$$




Quaternions = noncommutative associative 4-dimensional division algebra.

# Octonions

Hamilton wrote a letter to J. Graves, who went further:

$$\mathbb{O} = \mathbb{H} + \mathbb{H}v, v^2 = -1, \qquad \underline{\text{Octonions}}$$

8-dimensional, but not associative.

A. Cayley read Hamilton's paper and also discovered $\mathbb{O}$:
Octonions or Cayley numbers.

$\mathbb{R} \to \mathbb{C} \to \mathbb{H} \to \mathbb{O} \cdots$

Kervaire - Milnor (1958), finite dimensional division algebras$/\mathbb{R}$:
dimensions 1,2,4,8.

## Infinite Rings

Abstract Revolution led by D. Hilbert and E. Noether.

# Rings

H.M. Wedderburn (1908):

$\dim A < \infty$, no ideals $I \neq (0), I^2 = (0)$. Then

$$A \cong M_{n_1}(D_1) \oplus \cdots \oplus M_{n_r}(D_r),$$

$D_i$ division algebras.

What about $D_i$'s?

Albert, Brauer, Hasse, Noether:

Division algebras over algebraic number fields.

Arbitrary fields?

# Rings

Amitsur (1975): generic division algebras.

Merkuriev - Souslin (1984)

$F \ni$ all roots of 1

$D \sim D_1 \otimes \cdots \otimes D_r$,

$D_i$ generalized quaternion algebras.

# Rings

Hilbert (1888) : solved an important problem on invariants in a very strange way.

Gordan : “Das ist nicht Mathematik. Das ist Theologie.”

E. Noether (1920s) : theory of ideals in rings + 1932 ICM Plenary Talk.

Manifesto of Abstract Algebra.

Axiomatic Method: The World of Mathematics is seduced.

# Axiomatics

H. Weyl: "Nice general concepts do not fall into our laps by themselves. But definite concrete problems were first conquered in their undivided complexity, singlehanded by brute force, so to speak. Only afterwards the axiomaticians came along and stated: Instead of breaking the door with all your might and bruising your hands, you should have constructed such and such a key of skill, and by it you would have been able to open the door quite smoothly. But they can construct the key only because they are able, after the breaking in was successful, to study the lock from within."

Hermann Weyl

# Axiomatics

Mathematics is not a study of Axioms

(1) We formulate the most important properties of an object that are relevant to our study,
(2) We take these properties as axioms and study all objects that satisfy these axioms.

Since Noether, Mathematics speaks the language of Axioms. Even critics of Axiomatics write their papers in this way. It is like complaining about exceeding predominance of English...in English.

# Algebras

N. Jacobson, I. Kaplansky:

Structure theory of infinite dimensional noncommutative algebras.

Links to Operator Algebras.

Influence of Mathematical Physics:

everything becomes Infinite Dimensional and Noncommutative.

## Lie Algebras

Sophus Lie:

$G$ Lie group $\to T_1G$, $[\ ,\ ]$

$$\begin{cases} [x,y] = -[y,x] \\ [[x,y],z] + [[y,z],x] + [[z,x],y] = 0 \end{cases}$$

W. Killing, E. Cartan

$L$ has no ideals $I \neq (0), [I,I] = (0)$

$\Rightarrow L = L_1 \oplus \cdots \oplus L_s$,

$L_i$ simple, $A_n, B_n, C_n, D_n, G_2, F_4, E_6, E_7, E_8$.

# Lie Algebras

Building blocks for $G_2, F_4, E_6, E_7, E_8$:

nonassociative algebras

$\mathbb{O}$, $J = H_3(\mathbb{O})$ Jordan algebra

$G_2 = \operatorname{Der}\mathbb{O}$, $F_4 = \operatorname{Der} J$, etc.

30 - 40s H.Weyl : LIE ALGEBRAS

$$\Downarrow$$

influence on classification of finite simple groups

50s Chevalley Groups

Finite Fields

## Infinite Dimensional Lie Algebras

V. Kac, R. Moody, 1968

$\mathfrak{g}$ simple, $\dim_F \mathfrak{g} < \infty$

$L = \mathfrak{g} \otimes F[t^{-1}, t] + F\not{c}$

$L = \sum_{i \in \mathbb{Z}} L_i, \; V = \sum_{i \in \mathbb{Z}} V_i$

graded module, graded dimension

$\sum_i (\dim V_i) q^i$

characters $=$ series in $q$.

# Algebras

Combinatorial identities

Mathematical Physics

Vertex Operator Algebras

$V = V_{-1} + V_0 + V_1 + \cdots$

Aut $V$ = Monster Moonshine Representation

(Tits, Kac, Borcherds, Frenkel-Lepovsky-Meurman)

New hopes for the Classification

$J_1$ Janko group?

## 21st Century

1. New approaches to THE CLASSIFICATION.

2. New Noncommutative and Infinite Dimensional World.

3. Linear Algebra.

# 21st Century

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = a_1 \\ \vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n = a_n \end{cases}$$

Gauss Algorithm $\Rightarrow\approx n^3$ operations

Suppose that the size $n$ is HUGE, and, in addition, we don't know the coefficients precisely. But (!) most $a_{ij} = 0$.

Problem: do better than $n^3$.

# 21st Century

Many problems with Big Data and AI boil down to this problem.

Deep Learning = pattern recognition in multidimensional spaces.

Experience: inequality

$$a_1 x_1 + \cdots + a_m x_m \leq a$$

Many of them!

"Unreasonable effectiveness of Mathematics" E. Wigner