Some results on Kripke models over an arbitrary fixed frame

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Abstract

We study the relations of being substructure and elementary substructure between Kripke models of intuitionistic predicate logic with the same arbitrary frame. We prove analogues of Tarski's test and Löwenheim-Skolem's theorems as determined by our definitions. The relations between corresponding worlds of two Kripke models $\mathcal{K} \preceq \mathcal{K}'$ are studied.

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1 Introduction

In this paper we develop basic model theory for Kripke models of intuitionistic predicate logic. Some works have been already done in this direction. For example, in [9], a substructure of a given Kripke model was defined to be the result of restricting its frame and keeping the structures assigned to the remaining nodes the same. It was proved that, regarding this notion of substructure, the class of formulas of intuitionistic predicate logic that are preserved under taking substructure is the class of semipositive formulas, i.e., formulas such that each implicational subformula of them has an atomic antecedent.

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In [9], it was also noted that one may define substructure of a given Kripke structure to be a Kripke structure with the same frame, where the structures assigned to the nodes are (classical) substructures of the structures assigned to the corresponding nodes of the original model. We will use this definition.

2 Preliminaries

Let us fix an arbitrary language L. We use the usual Kripke semantics for intuitionistic theories based on L. Below, we briefly mention the definition of Kripke models.

A Kripke structure $\mathcal{K}$ for a language L, is a pair $\mathcal{K} = ((M_\alpha)_{\alpha \in K}, \leq)$ such that $(K, \leq)$ is a partially ordered set (called the frame of $\mathcal{K}$) and to each element (called a node) α of K is attached a classical structure M_α for L in which the interpretation of equality is an equivalence relation which may properly extend the true equality. For any two nodes α, β, if β is accessible from α (that is $\alpha \leq \beta$), then the world at α must be a weak substructure of the one at β. This means that M_β preserves truth in M_α of atomic sentences in L_α (the expanded language by adding constants for elements of M_α). So, tuples of elements of M_α may acquire new atomic properties, perhaps equality, in M_β.

Definition 2.1 (Forcing) *The forcing relation* $\Vdash$ *is defined between nodes and* L_α*-sentences inductively as follows:*

- *For atomic* φ, $M_\alpha \Vdash \varphi$ *if and only if* $M_\alpha \vDash \varphi$, *also,* $M_\alpha \nVdash \bot$;
- $M_\alpha \Vdash \varphi \vee \psi$ *if and only if* $M_\alpha \Vdash \varphi$ *or* $M_\alpha \Vdash \psi$;
- $M_\alpha \Vdash \varphi \wedge \psi$ *if and only if* $M_\alpha \Vdash \varphi$ *and* $M_\alpha \Vdash \psi$;
- $M_\alpha \Vdash \varphi \rightarrow \psi$ *if and only if for all* $\beta \geq \alpha$, $M_\beta \Vdash \varphi$ *implies* $M_\beta \Vdash \psi$;
- $M_\alpha \Vdash \forall x \varphi(x)$ *if and only if for all* $\beta \geq \alpha$ *and all* $a \in M_\beta, M_\beta \Vdash \varphi(a)$;
- $M_\alpha \Vdash \exists x \varphi(x)$ *if and only if there exists* $a \in M_\alpha$ *such that* $M_\alpha \Vdash \varphi(a)$.

By $M_\alpha \Vdash \varphi(\overline{x})$, one means $M_\alpha \Vdash \forall \overline{x} \varphi(\overline{x})$. $\neg\varphi$ is defined as $\varphi \rightarrow \bot$. One says that α decides φ whenever $M_\alpha \Vdash PEM_\varphi$, where PEM_φ is the sentence $\forall \overline{x}(\varphi(\overline{x}) \vee \neg\varphi(\overline{x}))$.

If a Kripke model decides atomic formulas, then the worlds of the model can be considered as normal structures, i.e. interpret "=" as the real equality and the relation between each world and each accessible world form it, is substructure (see again [9]).

For the standard model theoretic theorems and arguments we will use, we refer to any standard text book in model theory, e.g. Hodges [6]. We also refer to Van Dalen [4] for the basic concepts and results in intuitionistic logic and the theory of Kripke models.

3 Definitions and basic results

In this section we define the substructure and elementary substructure relations for Kripke models. We also prove some basic facts about these definitions.

Definition 3.1 *Let $\mathcal{K} = ((M_\alpha)_{\alpha\in K}, \leq)$ and $\mathcal{K}^{'} = ((M^{'}_\alpha)_{\alpha\in K}, \leq)$ be two Kripke models (with the same frame K) and for each $\alpha \in K$, M_α be a subset of $M^{'}_\alpha$. Suppose also that for any atomic formula $\varphi(\overline{x})$, $\alpha \in K$ and $\overline{a} \in M_\alpha$, $M_\alpha \Vdash \varphi(\overline{a})$ if and only if $M'_\alpha \Vdash \varphi(\overline{a})$. In this case we say that $\mathcal{K}$ is a substructure of $\mathcal{K}^{'}$ and denote it by $\mathcal{K} \subseteq \mathcal{K}'$.*

Note that $\mathcal{K} \subseteq \mathcal{K}'$ means that M_α is a substructure of M'_α for each α. Also, if both Kripke models force PEM_{atomic}, then $\mathcal{K} \subseteq \mathcal{K}'$ implies that $M_\alpha \Vdash \varphi(\overline{a})$ if and only if $M'_\alpha \Vdash \varphi(\overline{a})$ for any open formula φ. It is reasonable to choose other definitions for this notion. We prefer this since it makes developing Kripke model theory more similar to the classical one.

Definition 3.2 *Let $\mathcal{K}$ and $\mathcal{K}'$ be as above. We say $\mathcal{K}$ is an elementary substructure of $\mathcal{K}'$, denoted $\mathcal{K} \preceq \mathcal{K}'$, if:*

i) $\mathcal{K} \subseteq \mathcal{K}'$.

ii) For any formula $\varphi(\overline{x})$, $\alpha \in K$ and $\overline{a} \in M_\alpha$, $M_\alpha \Vdash \varphi(\overline{a})$ if and only if $M'_\alpha \Vdash \varphi(\overline{a})$.

Note that $\mathcal{K} \preceq \mathcal{K}'$ implies that the corresponding nodes in these Kripke models force the same sentences of the language.

Below, we give a version of Tarski's test in this situation.

Theorem 3.3 *Let $\mathcal{K} \subseteq \mathcal{K}'$ be two Kripke models. Then $\mathcal{K} \preceq \mathcal{K}'$ if and only if the following conditions hold for any formula $\varphi(x, \overline{y})$, $\alpha \in K$ and $\overline{a} \in M_\alpha$:*

i) If there is $b \in M'_\alpha$ such that $M'_\alpha \Vdash \varphi(b, \overline{a})$, then there is $c \in M_\alpha$ such that $M'_\alpha \Vdash \varphi(c, \overline{a})$.

ii) If there is $b \in M'_\alpha$ such that $M'_\alpha \nVdash \varphi(b, \overline{a})$, then there is $\beta \geq \alpha$ and $c \in M_\beta$ such that $M'_\beta \nVdash \varphi(c, \overline{a})$.

Proof Clearly, if $\mathcal{K} \preceq \mathcal{K}'$ then condition (i) holds. Now, suppose the assumption of condition (ii) holds. Since $M'_\alpha \nVdash \forall x\varphi(x, \overline{a})$ we have $M_\alpha \nVdash \forall x\varphi(x, \overline{a})$ and so there are $\beta \geq \alpha$ and $c \in M_\beta$ such that $M_\beta \nVdash \varphi(c, \overline{a})$. Hence $M'_\beta \nVdash \varphi(c, \overline{a})$.

Now, assuming (i) and (ii), we prove $\mathcal{K} \preceq \mathcal{K}'$ by induction on the complexity of formulas. We heavily use the assumption that the Kripke models have the same frame. The claim is true for atomic formulas by definition. In the induction step, we just check the case $\forall$. The others are trivial.

$\forall$: Suppose $M_\alpha \Vdash \forall x\varphi(x, \overline{a})$, $\overline{a} \in M_\alpha$, but $M^{'}_\beta \nVdash \varphi(b, \overline{a})$ for some $\beta \geq \alpha$ and $b \in M^{'}_\beta$. By assumption (ii), there are $\gamma \geq \beta$ and $c \in M_\gamma$ such that $M'_\gamma \nVdash \varphi(c, \overline{a})$ and so by induction hypothesis, $M_\gamma \nVdash \varphi(c, \overline{a})$ which is a contradiction. The other direction is obvious. $\square$

Example Let M_0, M and N be classical structures in the same language. Let M be an elementary substructure of N, and M_0 can be embedded in M. Then the Kripke model obtained by putting M above M_0 is an elementary substructure of the Kripke model obtained by putting N above M_0.

4 Löwenheim-Skolem type theorems for Kripke models

In this section, using theorem 3.3, we prove suitable versions of the downward and upward Löwenheim-Skolem theorems for Kripke models. First the downward one.

Let $\mathcal{K} = ((M_\alpha)_{\alpha \in K}, \leq)$ be a Kripke Model. Let $X_\alpha \subseteq M_\alpha$, $\alpha \in K$ and $X_\alpha \subseteq X_\beta$ whenever $\alpha \leq \beta$. In this case we call the family $X = \{X_\alpha\}_{\alpha \in K}$ a subset of $\mathcal{K}$ and write $X \subseteq \mathcal{K}$. Below, by a K-sequence of cardinals we mean a sequence $\overline{\kappa} = \{\kappa_\alpha\}_{\alpha \in K}$ of cardinals $\kappa_\alpha \geq |L| + \aleph_0$ such that $\kappa_\alpha \leq \kappa_\beta$ whenever $\alpha \leq \beta$.

Theorem 4.1 *Let $X \subseteq \mathcal{K}$ and $\overline{\kappa}$ be a K-sequence of cardinals. Assume for every $\alpha \in K$, $|X_\alpha| \leq \kappa_\alpha \leq |M_\alpha|$. Then there exists an elementary submodel $\mathcal{K}' \preceq \mathcal{K}$ such that $X_\alpha \subseteq M'_\alpha$ and $|M'_\alpha| = \kappa_\alpha$ for any α.*

Proof Without loss of generality, we assume that X_α contains all constants and its cardinality is κ_α for any node α. This can be done by adding new elements to X_α from M_α, if necessary.

We define an increasing sequence X_n of subsets of $\mathcal{K}$ as follows. Take $X_0 = X$ and let $X_n = \{X_{\alpha,n}\}_{\alpha \in K}$ be defined. We define $X_{n+1} = \{X_{\alpha,n+1}\}_{\alpha \in K}$ as follows. For each formula $\varphi(x, \overline{a})$ where $\overline{a} \in X_{\alpha,n}$, choose b_φ such that $M_\alpha \Vdash \varphi(b_\varphi, \overline{a})$, and choose $c_\varphi \in M_\alpha$ such that $M_\alpha \nVdash \varphi(c_\varphi, \overline{a})$. In case there is no such b_φ, let b_φ be a fixed element of X_α. Do the same for c_φ. Define $Y_{\alpha,n+1} = \bigcup_\varphi \{b_\varphi, c_\varphi\}$ where φ ranges over all formulas with parameters in $X_{\alpha,n}$. Now, let $X_{\alpha,n+1} = X_{\alpha,n} \cup (\bigcup_{\beta \leq \alpha} Y_{\beta,n+1})$. Clearly, X_{n+1} is a subset of $\mathcal{K}$.

Put $M'_\alpha = \bigcup_n X_{n,\alpha}$. Considering suitable choices for φ shows that M'_α is a submodel of M_α. For example, for any function symbol F and tuple $\overline{a} \in X_{\alpha,n}$, there is a (unique) $b \in X_{\alpha,n+1}$ such that $F(\overline{a}) = b$. We have also $|M'_\alpha| = \kappa_\alpha$. To see this, note that we have assumed $|X_\alpha| = \kappa_\alpha$ and $\kappa_\alpha \geq |L|$.

We show that $\mathcal{K} = ((M'_\alpha)_{\alpha \in K}, \leq)$ is an elementary submodel of $\mathcal{K}$ by using 3.3. Let $\overline{a} \in M'_\alpha$ and $\varphi(x, \overline{y})$ be a formula. There is an n such that $\overline{a} \in X_{\alpha,n}$. By construction, if there is a $b \in M_\alpha$ such that $M_\alpha \Vdash \varphi(b, \overline{a})$ then there is a $b' \in X_{\alpha,n+1} \subseteq M'_\alpha$ such that $M_\alpha \Vdash \varphi(b', \overline{a})$. Similarly, if $M_\alpha \nVdash \varphi(b, \overline{a})$ where $\overline{a} \in X_{\alpha,n}$, then there is $b' \in X_{\alpha,n+1} \subseteq M'_\alpha$ such that $M_\alpha \nVdash \varphi(b', \overline{a})$. □

Now, we give a proof of the upward Löwenheim-Skolem theorem by using ultraproducts. This construction will work only for Kripke Models over a special kind of frames. We leave open the question in the general case. Ultraproducts of Kripke models were studied by D. Gabbay (see [5]). One of the differences between two approaches is that we only consider ultraproducts of Kripke models over the same frame. Also, according to our definition, ultraproduct of a family of Kripke models will have the same frame as the frame of the members of the family.

Let K be a partially ordered set with the property that for each $\alpha \in K$, the set of elements above α is finite. In the rest of this section we assume that all Kripke models we consider have frames with this property. Let $\mathcal{K}^i, i \in I$ be an indexed family of Kripke models over K and U be an ultrafilter over I. Using the classical construction we define $M_\alpha = \prod_U M_\alpha^i$. Clearly, M_α is a weak substructure of M_β whenever $\alpha \leq \beta$. Let $\prod_U \mathcal{K}^i$ denote the Kripke model $((M_\alpha)_{\alpha \in K}, \leq)$.

Theorem 4.2 *For any formula $\varphi(\overline{x})$, any α and any finite tuple $([a_i], ...)$ of elements in M_α, $M_\alpha \Vdash \varphi([a_i], ...)$ if and only if $\{i : M_\alpha^i \Vdash \varphi(a_i, ...)\} \in U$.*

Proof Induction on the complexity of formulas. The claim is true for atomic formulas by the fact that forcing and satisfaction are the same for them. The cases $\vee, \wedge$ and $\exists$ are clear. We check the cases $\rightarrow$ and $\forall$.

$\rightarrow$: Let $\{i : M_\alpha^i \Vdash (\varphi \rightarrow \psi)(a_i, ...)\} \in U$. This means

$$\{i : \quad \forall \beta \geq \alpha, \quad M_\beta^i \Vdash \varphi \Longrightarrow M_\beta^i \Vdash \psi\} \in U.$$

By the assumption on K, this is equivalent to

$$\forall \beta \geq \alpha, \quad \{i : \quad M_\beta^i \Vdash \varphi \Longrightarrow M_\beta^i \Vdash \psi\} \in U.$$

Since U is an ultrafilter, the last statement is equivalent to:

$$\forall \beta \geq \alpha, \quad \{i : \quad M_\beta^i \Vdash \varphi(a_i, ...)\} \in U \Longrightarrow \{i : \quad M_\beta^i \Vdash \psi(a_i, ...)\} \in U.$$

By induction hypothesis this is equivalent to:

$$\forall \beta \geq \alpha, \quad M_\beta \Vdash \varphi([a_i], ...) \Longrightarrow M_\beta \Vdash \psi([a_i], ...)$$

which means that $M_\alpha \Vdash (\varphi \rightarrow \psi)([a_i], ...)$.

$\forall$: Let $\{i : \quad M_\alpha^i \Vdash \forall x \varphi(x, a_i, ...)\} \in U$. Then

$$\{i : \quad \forall \beta \geq \alpha, \forall c_i \in M_\beta^i, \ M_\beta^i \Vdash \varphi(c_i, a_i, ...)\} \in U.$$

By the assumption on K, this is equivalent to:

$$\forall \beta \geq \alpha, \ \{i : \quad \forall c_i \in M_\beta^i, \ M_\beta^i \Vdash \varphi(c_i, a_i, ...)\} \in U.$$

It is not hard to see that the above statement is equivalent to

$$\forall \beta \geq \alpha, \ \forall [c_i] \in M_\beta, \ \{i : \quad M_\beta^i \Vdash \varphi(c_i, a_i, ...)\} \in U.$$

But, by induction hypothesis, this is equivalent to

$$\forall \beta \geq \alpha, \ \forall [c_i] \in M_\beta, \ M_\beta \Vdash \varphi([c_i], [a_i], ...)$$

i.e. $M_\alpha \Vdash \forall x \varphi(x, [a_i], ...)$. $\square$

The following can be proved similar to the corresponding case in classical model theory.

Corollary 4.3 *A theory T has a Kripke model over a frame K (of the above form) if and only if each finite part of it has a Kripke model over K.*

Now we prove an upward Löwenheim-Skolem theorem for Kripke models (over the frames with the above property).

An ultrafilter U over an infinite cardinal λ is said to be λ-regular if there exists a set $E \subseteq U$ of power $|E| = \lambda$ such that each $i \in \lambda$ belongs to only finitely many $e \in E$. It is known that such ultrafilters always exist (see [3], P. 248-249).

Theorem 4.4 *Let $\mathcal{K}$ be a Kripke model all of whose worlds are infinite structures. Let also $\overline{\kappa}$ be a K-sequence of cardinals such that $\kappa_\alpha \geq |M_\alpha| + |L|$. Then there exists a Kripke model $\mathcal{K}'$ such that $\mathcal{K} \preceq \mathcal{K}'$ and $|M'_\alpha| = \kappa_\alpha$ for any α.*

Proof Let $\lambda = sup\ \{\kappa_\alpha : \alpha \in K\}$ and U be a λ-regular ultrafilter over λ. Using [3], Prop. 4.3.7, one can see that $|\prod_U M_\alpha| \geq \aleph_0^\lambda > \lambda$. By Theorem 4.2, $\prod_U \mathcal{K}$ can be viewed as an elementary extension of $\mathcal{K}$. Now applying Theorem 4.1 to $\prod_U \mathcal{K}$, one obtains an elementary extension of $\mathcal{K}$ with the desired cardinality properties.$\square$

5 How extending a Kripke model affects its worlds

In this section we study the relation between corresponding worlds of two Kripke models of the form $\mathcal{K} \preceq \mathcal{K}'$. We present three propositions in this respect. However, we leave the complete description as a question.

In the sequel, we assume that all Kripke models we consider decide atomic formulas. So, we can assume that the accessability relation between the nodes is substructure. For a linear Kripke model $\mathcal{K}$, $\bigcup \mathcal{K}$ denotes the (standard) union of the structures in $\mathcal{K}$.

Proposition 5.1 *Let $\mathcal{K} \preceq \mathcal{K}'$ be linear. Then $\bigcup \mathcal{K} \preceq \bigcup \mathcal{K}'$.*

Proof It was proved in [7], using induction on formulas, that if α is a node in a linear Kripke model and φ is an $\exists$-free formula, then $M_\alpha \Vdash \varphi$ if and only if the union of the worlds above α satisfies φ. Now let $\psi(\overline{x})$ be a formula. We have to show that for any α and $\overline{a} \in M_\alpha$, $\bigcup \mathcal{K} \models \psi(\overline{a})$ implies $\bigcup \mathcal{K}' \models \psi(\overline{a})$. Assume $\bigcup \mathcal{K} \models \psi(\overline{a})$. Let $\theta(\overline{x})$ be an $\exists$-free formula classically equivalent to $\psi(\overline{x})$. By the above mentioned fact, we have $M_\alpha \Vdash \theta(\overline{a})$. Therefore, $M'_\alpha \Vdash \theta(\overline{a})$. Hence $\bigcup \mathcal{K}' \models \theta(\overline{a})$. So $\bigcup \mathcal{K}' \models \psi(\overline{a})$. $\square$

For any theory T, $\mathcal{H}(T)$ is the intuitionistic theory of all T-normal Kripke structures. This theory was introduced and axiomatized in [2]. $\mathcal{H}(T)$ is an intuitionistic theory which is closed under the Friedman translation (see [8], and [1] Page 217).

Proposition 5.2 *Let $\mathcal{K} \preceq \mathcal{K}'$ have a finite-depth frame or have ω as frame. Let T be a classical theory. We have $M_\alpha \models T$ for all α if and only if $M'_\alpha \models T$ for all α.*

Proof Let $\mathcal{K} \preceq \mathcal{K}'$ be finite-depth or ω-framed and for every node α, $M_\alpha \models T$. Hence $\mathcal{K} \Vdash \mathcal{H}(T)$. Therefore each finite-depth or ω-framed Kripke model of it, including $\mathcal{K}'$, is T-normal, see [10] for a proof of this fact in case of Heyting arithmetic. The proof actually works for any intuitionistic theory which decides atomic formulas and is closed under the Friedman translation. So $M'_\alpha \models T$ for any α . $\square$

Let us mention the first pruning lemma, see [10]. It says that if β is a node of a Kripke model $\mathcal{K}$, φ and ψ are formulas in L_β such that no free variables of ψ are bound in φ and $\beta \nVdash \psi$, then $\beta \Vdash \varphi^\psi$ if and only if $\beta \Vdash^\psi \varphi$. Here φ^ψ is the Friedman translation of φ by ψ and $\Vdash^\psi$ denotes forcing in the Kripke structure $\mathcal{K}^\psi$ obtained from the original one by pruning away nodes forcing ψ.

Proposition 5.3 *Let $\mathcal{K} \preceq \mathcal{K}'$ have a two-node frame $(\{0,1\}, \leqslant)$. Then for $\alpha = 0, 1$ we have, $M_\alpha \preceq M'_\alpha$.*

Proof Obviously, terminal nodes have the required property since forcing and satisfaction for them are equivalent. Assume $M_0 \npreceq M'_0$. Note that any formula is classically equivalent to a semipositive one and forcing of a semipositive formula in a node implies its satisfaction in the (world attached to the) node. Let $\varphi(\overline{a})$, $\overline{a} \in M_0$, be semipositive and $M_0 \vDash \varphi(\overline{a})$ but $M'_0 \nvDash \varphi(\overline{a})$. Hence $M'_0 \nVdash \varphi(\overline{a})$. Consider the following two cases. Case 1: $M'_1 \nVdash \varphi(\overline{a})$. Hence $M'_0 \Vdash \mathcal{H}(\neg\varphi(\overline{a}))$. Hence $M_0 \Vdash \mathcal{H}(\neg\varphi(\overline{a}))$. But each finite Kripke model of $\mathcal{H}(\neg\varphi(\overline{a}))$ is $\neg\varphi(\overline{a})$-normal, so we get a contradiction. Case 2: $M'_1 \Vdash \varphi(\overline{a})$. Use pruning with respect to $\varphi(\overline{a})$ to delete maximal nodes in two Kripke models. Using the first pruning lemma it is easy to see that the two one-node Kripke models M_0 and M'_0 obtained are still of the form $M_0 \preceq M'_0$. $\square$

The following example shows that the converse of the previous proposition is not true.

Example Let $M \subseteq N$ be two classical structures and $M \npreceq N$. Let $M_0 \preceq N$ and $M_0 \preceq M$. Now consider two Kripke models $\mathcal{K}$: M_0 above M_0 and $\mathcal{K}'$: N above M. This provides us with Kripke models $\mathcal{K} \subseteq \mathcal{K}'$ such that the relation between each two corresponding worlds of them is elementary extension but $\mathcal{K} \npreceq \mathcal{K}'$. In fact, in this case, $\mathcal{K}$ and $\mathcal{K}'$ are not elementarily equivalent. To see why this last statement is true, note that the first Kripke model forces the sentence PEM_φ, for any formula φ, while this is not the case for the second one. To have a more concrete example, one can consider two models M, N elementarily equivalent to $\mathbb{N}$ such that $M \npreceq N$ and the standard model $\mathbb{N}$ as M_0, which is elementarily embedded in both of them.

Question If $\mathcal{K} \preceq \mathcal{K}'$, what can we say in general about the corresponding worlds in them? Is the relation between the worlds is elementary substructure?

We end the paper with an easy proposition which states an intuitionistic version of a quantifier elimination test in the classical logic.

Proposition 5.4 *Let T be an intuitionistic theory deciding atomic formulas. Suppose that every formula of the form $\exists x\psi(x, \overline{y})$, where $\psi(x, \overline{y})$ is quantifier-free , is T-equivalent to a quantifier-free formula . Then T has quantifier-elimination, i.e., any formula in T is equivalent to a quantifier-free formula.*

Proof Use induction on formulas. The cases $\vee$, $\wedge$ and $\rightarrow$ are obvious.

$\exists$: Let $T \vdash \varphi(x, \overline{y}) \leftrightarrow \psi(x, \overline{y})$ where ψ is quantifier-free. So, we have $T \vdash \exists x \varphi(x, \overline{y}) \leftrightarrow \exists x \psi(x, \overline{y})$. By the assumption, $\exists x \psi(x, \overline{y})$ is T-equivalent to a quantifier-free formula. Therefore, $\exists x \varphi(x, \overline{y})$ is T-equivalent to a quantifier-free formula.

$\forall$: Let $T \vdash \varphi(x, \overline{y}) \leftrightarrow \psi(x, \overline{y})$ where ψ is quantifier-free. So, we have $T \vdash \forall x \varphi(x, \overline{y}) \leftrightarrow \forall x \psi(x, \overline{y})$. Using decidability of atomic formulas, we get the following intuitionistic equivalences:

$$\forall x \psi(x, \overline{y}) \equiv \forall x \neg\neg \psi(x, \overline{y}) \equiv \neg \exists x \neg \psi(x, \overline{y}).$$

Now, using the assumption, we obtain the result. $\Box$

Corollary 5.5 *Under the assumptions of the above proposition, $T \vdash PEM$.*

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