Comparing Constructive Arithmetical Theories Based On NP-PIND and coNP-PIND

Morteza Moniri

Abstract

In this note we show that the intuitionistic theory of polynomial induction on Π_1^{b+}-formulas does not imply the intuitionistic theory IS_2^1 of polynomial induction on Σ_1^{b+}-formulas. We also show the converse assuming the Polynomial Hierarchy does not collapse. Similar results hold also for length induction in place of polynomial induction. We also investigate the relation between various other intuitionistic first-order theories of bounded arithmetic. Our method is mostly semantical, we use Kripke models of the theories.

2000 Mathematics Subject Classification: 03F30, 03F55, 03F50, 68Q15.

Key words and phrases:

Intuitionistic Bounded Arithmetic, Polynomial Hierarchy, Polynomial Induction, Length Induction, NP-formulas, coNP-formulas, Kripke Models.

0 Introduction

In [B1], Buss introduced some particular first-order theories of bounded arithmetic. The language of these theories extends the usual language of arithmetic by adding function symbols $\lfloor \frac{x}{2} \rfloor$ ($= \frac{x}{2}$ rounded down to the nearest integer), $|x|$ (=the number of digits in the binary notation for x) and $\#$ ($x\#y = 2^{|x||y|}$). The set BASIC of basic axioms for the theories of bounded arithmetic is a finite set of (universal closures of) quantifier-free formulas expressing basic properties of the relations and functions of the language.

The set of sharply bounded formulas is the set of bounded formulas which all quantifiers occurring in them are sharply bounded quantifiers, i.e. of the form $\exists x \leqslant |t|$ or $\forall x \leqslant |t|$ where t is a term not involving x.

Following Buss [B1], we define a hierarchy of bounded formulas:

(1) $\Sigma_0^b = \Pi_0^b$ is the set of all sharply bounded formulas.

(2) Σ_{i+1}^b is defined inductively by:

(2a) $\Pi_i^b \subseteq \Sigma_{i+1}^b$;

(2b) If $A \in \Sigma_{i+1}^b$, so are $(\exists x \leqslant t)A$ and $(\forall x \leqslant |t|)A$;

(2c) If $A, B \in \Sigma^b_{i+1}$, so are $A \wedge B$ and $A \vee B$;

(2d) If $A \in \Sigma^b_{i+1}$ and $B \in \Pi^b_{i+1}$, then $\neg B$ and $B \to A$ are in Σ^b_{i+1}.

(3) Π^b_{i+1} is defined inductively as follows:

(3a) $\Sigma^b_i \subseteq \Pi^b_{i+1}$;

(3b) If $A \in \Pi^b_{i+1}$, so are $(\forall x \leqslant t)A$ and $(\exists x \leqslant |t|)A$;

(3c) If $A, B \in \Pi^b_{i+1}$, so are $A \wedge B$ and $A \vee B$;

(3d) If $A \in \Pi^b_{i+1}$ and $B \in \Sigma^b_{i+1}$, then $\neg B$ and $B \to A$ are in Π^b_{i+1}.

(4) Σ^b_{i+1} and Π^b_{i+1} are the smallest sets which satisfy (1)-(3).

Note that by the above definition, negation of a Π^b_1-formula is a Σ^b_1-formula. This is an important point when we work with intuitionistic theories. The Σ^b_1-formulas represent exactly the NP-relations in the standard model. For this reason, they are also called NP-formulas.

The most important theory among the theories of bounded arithmetic is S^1_2, obtained by adding the scheme $PIND$ for Σ^b_1-formulas to $BASIC$:

$(A(0) \wedge \forall x(A(\llcorner \frac{x}{2} \lrcorner) \to A(x))) \to \forall x A(x)$

The main reason is the following theorem. Note that a function f is said to be Σ^b_1-definable in S^1_2 if and only if it is provably total in S^1_2 with a Σ^b_1-formula defining the graph of f.

Theorem 0.1 (Buss, [B1]) A function is Σ^b_1-definable in S^1_2 if and only if it is polynomial time computable.

The schemes $LIND$ and IND are

$(A(0) \wedge \forall x(A(x) \to A(x+1))) \to \forall x A(|x|)$ and

$(A(0) \wedge \forall x(A(x) \to A(x+1))) \to \forall x A(x)$, respectively.

The following theorem will be used throughout this paper.

Theorem 0.2 The following theories are equivalent to S^1_2:

(1) $BASIC + \Sigma^b_1 - LIND$

(2) $BASIC + \Pi^b_1 - PIND$

(3) $BASIC + \Pi^b_1 - LIND$

We also have $BASIC + \Sigma^b_1 - IND \equiv BASIC + \Pi^b_1 - IND \vdash S^1_2$.

Proof See [B1] and [B4]. $\square$

The theory IS^1_2 is the intuitionistic theory axiomatized by $BASIC$ plus the scheme $PIND$ on positive Σ^b_1 formulas (denoted Σ^{b+}_1), i.e. Σ^b_1-formulas which do not contain $\neg$

and $\rightarrow$. This theory was introduced and studied by Cook and Urquhart and by Buss (see [CU] and [B3]). A function f is Σ_1^{b+}-definable in IS_2^1 if it is provably total in IS_2^1 with a Σ_1^{b+}-formula defining the graph of f. The most important theorem about IS_2^1 they proved is this:

Theorem 0.3 (Cook and Urquhart, [CU])

(i) If f is a polynomial time computable function then f is Σ_1^{b+}-definable in IS_2^1.

(ii) If $IS_2^1 \vdash \forall \overline{x} \exists y \phi(\overline{x}, y)$ then there is a polynomial time computable function f such that $IS_2^1 \vdash \forall \overline{x} \phi(\overline{x}, f(\overline{x}))$.

Note that, in part (ii) above, the symbol f in the formula does not belong to the given language; however by part (i), it can be expressed in our language.

A positive Π_1^b formula (denoted Π_1^{b+}), is also defined to be a Π_1^b-formula which does not contain $\neg$ and $\rightarrow$.

The theory PV is an equational theory of polynomial time functions introduced by Cook, PV_1 is its (conservative) extension to classical first-order logic and IPV is the intuitionistic theory of PV plus polynomial induction on NP formulas. Here an NP-formula is a formula equivalent to an atomic formula (in the language of PV) followed by a number of bounded existential quantifiers (see [CU]). The NP-formulas represent precisely the NP relations in the standard model. coNP-formulas are defined dually.

Our main results in this paper are that over a natural intuitionistic base theory (i.e., the intuitionistic deductive closure of $BASIC$), coNP induction does not imply NP induction; and that assuming the polynomial hierarchy does not collapse, neither does NP induction imply coNP induction. This is in sharp contrast to the case for classical logic, in which the two principles are equivalent.

1 Kripke models of intuitionistic bounded arithmetic

Here we briefly describe Kripke models. All theories we will study prove the principle of excluded middle PEM for atomic formulas and so we can use a slightly simpler version of the definition of Kripke models, see [V].

A Kripke structure for a language L can be considered as a set of classical structures for L partially ordered by the relation substructure. We can assume without loss of generality that this partially ordered set is a rooted tree. For every node α, L_α denotes the expansion of L by adding constants for elements of M_α. The forcing relation $\Vdash$ is defined inductively as follows:

- For atomic φ, $M_\alpha \Vdash \varphi$ if and only if $M_\alpha \models \varphi$, also, $M_\alpha \nVdash \bot$;
- $M_\alpha \Vdash \varphi \vee \psi$ if and only if $M_\alpha \Vdash \varphi$ or $M_\alpha \Vdash \psi$;
- $M_\alpha \Vdash \varphi \wedge \psi$ if and only if $M_\alpha \Vdash \varphi$ and $M_\alpha \Vdash \psi$;

- $M_\alpha \Vdash \varphi \to \psi$ if and only if for all $\beta \geq \alpha$, $M_\beta \Vdash \varphi$ implies $M_\beta \Vdash \psi$;
- $M_\alpha \Vdash \forall x\varphi(x)$ if and only if for all $\beta \geq \alpha$ and all $a \in M_\beta, M_\beta \Vdash \varphi(a)$;
- $M_\alpha \Vdash \exists x\varphi(x)$ if and only if there exists $a \in M_\alpha$ such that $M_\alpha \Vdash \varphi(a)$.

A Kripke model forces a formula $\varphi(\overline{x})$, if each of its nodes (equivalently its root) forces $\forall\overline{x}\varphi(\overline{x})$. A Kripke model is $BASIC$-normal if each node (world) of it satisfies $BASIC$. It is Δ_0^b-elementary if its accessible relation is Δ_0^b-extension, i.e. for any two nodes $\alpha \leq \beta$ and any Δ_0^b-formula $A(\overline{x})$ and $\overline{a} \in M_\alpha$, $M_\alpha \vDash A(\overline{b})$ if and only if $M_\beta \vDash A(\overline{b})$. It decides sharply bounded formulas if it forces the axiom PEM (that is, $\varphi \vee \neg\varphi$) restricted to sharply bounded formulas.

By $IBASIC$ we mean the intuitionstic theory axiomatized by $BASIC$ axioms.

Lemma 1.1 Kripke models of $IBASIC$ are exactly $BASIC$-normal Kripke models.

Proof Using the fact that atomic formulas are decidable in $IBASIC$ ([B3, Th.3]) and this theory is universal, the proof is straightforward. □

It is well-known and easy to prove that a Kripke model is Δ_0-elementary extension (that is, its accessible relation is Δ_0-extension) if and only if forcing and satisfaction of bounded formulas in each node (world) of it are equivalent if and only if, it decides bounded formulas. The following states a similar result for sharply bounded formulas.

Proposition 1.2 Suppose $\mathcal{K}$ is a $BASIC$-normal Kripke model. The following are equivalent:

(i) $\mathcal{K}$ is Δ_0^b-elementary extension.

(ii) Forcing and satisfaction of every sharply bounded formula in every node of $\mathcal{K}$ are equivalent.

(iii) $\mathcal{K}$ decides Δ_0^b-formulas.

Proof Induction on formulas. □

Let M and N be two models of $BASIC$. Let $Log(M) = \{a \in M : \exists b \in M$ $a \leq |b|\}$. N is a weak end extension of M if N extends M and $Log(N)$ is an end extension of $Log(M)$, i.e. for all $a \in Log(M)$, $b \in Log(N)$ with $N \vDash b \leq a$ we have $b \in Log(M)$. It is known and easy to see that weak end extensions are always Δ_0^b-elementary.

Corollary 1.3 We have

(i) Forcing and truth of sharply bounded formulas in each node of a weak end extension Kripke model of $IBASIC$ are equivalent.

(ii) Every Kripke model of IS_2^1 is Δ_0^b-elementary extension.

Proof Sharply bounded formulas are decidable in IS_2^1 ([CU]).□

Lemma 1.4 Suppose $\mathcal{K}$ is a weak end extension $BASIC$-normal Kripke model. Then for any node α in $\mathcal{K}$ and any Σ_1^{b+} L_α-sentence A we have, $M_\alpha \Vdash A$ if and only if $M_\alpha \models A$.

Proof Let A and $\mathcal{K}$ be as above. We use induction on the complexity of A to prove the desired property. For atomic formulas this is obvious by definition of forcing. In the induction step, there are four cases $\vee, \wedge$, bounded existential quantifier and sharply bounded universal quantifier. We just treat one part of the last case. The others are easy.

Let $\forall x \leq |t(\overline{a})| A(x, \overline{a})$ be a Σ_1^{b+} L_α-sentence. Let $M_\alpha \models \forall x \leq |t(\overline{a})| A(x, \overline{a})$. To prove $M_\alpha \Vdash \forall x \leq |t(\overline{a})| A(x, \overline{a})$, using the induction hypotheses, it is enough to show that this formula is satisfied in any node β above α. But this can be easily verified by the assumption that the Kripke model is a weak end extension Kripke model.$\Box$

A Kripke model is S_2^1-normal if each of its worlds satisfies S_2^1.

Theorem 1.5 Any S_2^1-normal weak end extension Kripke model forces IS_2^1.

Proof Using the definition of forcing, the proof is straightforward. However, we sketch the proof. Suppose $\mathcal{K}$ is an S_2^1-normal weak end extension Kripke model and $A(x, \overline{y}) \in \Sigma_1^{b+}$. Let M_α be a node in $\mathcal{K}$ such that it forces the assumptions of an instance of $PIND$ on a formula $A(x, \overline{b})$, where $\overline{b} \in M_\alpha$. We have to show that $M_\alpha \Vdash A(c, \overline{b})$ for any $c \in M_\alpha$. Using the above lemma, it is easy to see that $M_\alpha \models A(0, \overline{b}) \wedge \forall x (A(\llcorner \frac{x}{2} \lrcorner, \overline{b}) \rightarrow A(x, \overline{b}))$. Hence $M_\alpha \models A(c, \overline{b})$ for any $c \in M$, since $M_\alpha \models S_2^1$. So, $M_\alpha \Vdash A(c, \overline{b})$ for any $c \in M$. $\Box$

Lemma 1.6 Suppose $\mathcal{K}$ is a weak end extension $BASIC$-normal Kripke model. Then for any node α in $\mathcal{K}$ and any Π_1^{b+} L_α-sentence A we have, $M_\alpha \Vdash A \leftrightarrow \neg\neg A$.

Proof Induction on the complexity of formulas. $\Box$

Theorem 1.7 Any reversely well founded $BASIC$-normal weak end extension Kripke model whose terminal nodes model S_2^1 forces $BASIC + \Pi_1^{b+}$-PIND.

Proof It is known that S_2^1 (classically) proves $BASIC + \Pi_1^b$-PIND. Now, to complete the proof use these two general facts about Kripke models: (i) forcing and satisfaction of any formula in terminal nodes are equivalent, (ii) for any node α and formula φ, $\alpha \Vdash \neg\neg\varphi$ iff for each $\beta \geq \alpha$ there exists $\gamma \geq \beta$ such that $\gamma \Vdash \varphi$. $\Box$

Proposition 1.8 The union of the worlds in any linear weak end extension Kripke model of $BASIC + \Pi_1^{b+}$-PIND satisfies $BASIC + \Pi_1^{b+}$-PIND.

Proof Let A be a formula in which each instance of $\exists$ appears sharply bounded and $\mathcal{K}$ be a linear weak end extension Kripke model of $BASIC + \Pi_1^{b+}$-PIND. One can use induction on the complexity of A to show that A is forced in $\mathcal{K}$ if and only if the union of the worlds in $\mathcal{K}$ satisfies A. Now, to prove the Proposition, it is enough to note that any instance of $PIND$ on a Π_1^{b+} formula is of the mentioned form. $\Box$

2 $NP-$PIND versus coNP-PIND

In this section we use the basic results on Kripke models proved in Section 1 to compare the intuitionistic theories based on various schemes of induction on NP and $coNP$ formulas.

In the following theorem, we use [J1]. In [J1] and [J2], a model of the theory S_2^0 (the classical theory axiomatized by $BASIC$ plus $PIND$ on sharply bounded formulas) was constructed to witness a famous independence result of G. Takeuti [T], i.e. $S_2^0 \nvdash \forall x \exists y(x = 0 \vee x = y+1)$. Takeuti proved this result by use of a proof-theoretic method.

Theorem 2.1 The intuitionistic theory axiomatized by $BASIC + \Pi_1^{b+}$-PIND does not imply IS_2^1.

Proof If $M \subseteq N$ are models of $BASIC$ and N is a weak end extension of M, M is said to be length-initial in N by [J1]. Also, in [J1], for a special model $M \vDash S_2^1$ a substructure $M' \subseteq M$ is constructed such that M' is length-initial in M and the modified (restricted) subtraction $\dot{-}$ function is not provably total in M'. By Theorem 1.7, putting M above M' produces a Kripke model of $BASIC + \Pi_1^{b+}$-PIND. On the other hand, this Kripke model does not force IS_2^1. The reason is that S_2^1 is $\forall\Sigma_1^b$-conservative over IS_2^1 (see e.g., [A, Th. 3.17]) and so if the Kripke model forces IS_2^1, using forcing definition, its root would be a model of $\forall x, y \exists z \leq x(x \dot{-} y = z)$.$\square$

In the theory IPV which is the natural conservative extension of IS_2^1 to the language of PV, any Σ_1^{b+} formula is equivalent to an atomic formula (in the language of PV) followed by a number of bounded existential quantifiers (see [CU]).

Below, an NP formula is such a formula and a coNP formula is a PV-atomic formula followed by a number of bounded universal quantifiers. $\neg\neg NP$-formulas are doubly negated NP-formulas. The theory IPV can be axiomatized by PV plus $NP - PIND$, see [B2].

In general, the negative translation of a formula is obtained by replacing any subformula of the form $\psi \vee \eta$, resp. $\exists x \psi$, by $\neg(\neg\psi \wedge \neg\eta)$, resp. $\neg\forall x \neg\psi$ and inserting $\neg\neg$ in front of all atomic sub-formulas, except $\bot$. If $T \vdash_c \varphi$, then the set of negative translations of the formulas in T, intuitionistically proves the negative translation of φ, i.e. φ^-, see [TD].

In the following, the notation $\equiv_i$ between two sets of formulas is used to show that they have the same intuitionistic consequences. Also, $\vdash_i$ denotes provability in intuitionistic (first-order) logic.

Proposition 2.2 We have

(i) $PV+$ co$NP - PIND \equiv_i PV+$ co$NP - LIND$,

(ii)$PV + \neg\neg NP - PIND \equiv_i PV + \neg\neg NP - LIND$.

Proof We just prove the case (i). The other can be proved similarly.

First observe that the two theories are classically equivalent (see [B4] and note that in the presence of PV one has access to all polynomial time functions). Now, to obtain the desired intuitionistic equivalence, note that both intuitionistic theories are obviously closed under the negative translation. $\Box$

The replacement (bounded collection) axiom on a formula $\varphi(x,y)$ is:

$\forall x \leq |t| \exists y \leq s\varphi(x,y) \leftrightarrow \exists \omega \leq SqBd(s,t) \forall x \leq |t| (\beta(Sx,\omega) \leq s \wedge \varphi(x, \beta(Sx,\omega))$

where s and t are arbitrary terms and $SqBd(s,t)$ is a term which, roughly speaking, estimates the size of the sequence (s,t).

This axiom, which is called BB, enables us to interchange sharply bounded quantifiers with bounded quantifiers.

S_2^1 proves the above scheme for any Σ_1^b-formula φ (see [B1, Th. 2.7.14]).

Theorem 2.3 We have

(i) $PV+$ co$NP - IND \equiv_i PV + \neg\neg NP - IND$.

(ii) $PV+$ co$NP - LIND \equiv_i PV + \neg\neg NP - LIND$.

(iii) $PV+$ co$NP - PIND \equiv_i PV + \neg\neg NP - PIND$.

Proof (i) We argue informally in $PV+$ co$NP-IND$ and prove $PV+\neg\neg NP-IND$. Let $A(x,\overline{y})$ be atomic and assume:

(a) $\forall x(\neg\neg\exists \overline{y} \leqslant \overline{t} A(x,\overline{y}) \rightarrow \neg\neg\exists \overline{y} \leqslant \overline{t} A(x+1,\overline{y}))$ and

(b) $\neg\exists \overline{y} \leqslant \overline{t} A(a,\overline{y})$ for some (term) a.

Using (a) and (b), one obtains co$NP - IND$ on the formula $\forall z \leqslant a(x+z=a \rightarrow \forall \overline{y} \leqslant \overline{t} \neg A(z,\overline{y}))$ and so $\forall x \forall z \leqslant a(x+z=a \rightarrow \forall \overline{y} \leqslant \overline{t} \neg A(z,\overline{y}))$. Putting $x=a$, one gets $\neg\exists \overline{y} \leqslant \overline{t} A(0,\overline{y}))$. What we have done is proved $\neg\exists \overline{y} \leqslant \overline{t} A(0,\overline{y}))$ from (a) and (b). So, indeed, we proved the instance of IND on the formula $\neg\neg\exists \overline{y} \leqslant \overline{t} A(x,\overline{y})$.

Now we prove $PV + \neg\neg NP - IND \vdash_i PV+$ co$NP - IND$. Let $A(x,\overline{y})$ be atomic and assume:

(a) $\forall x(\forall \overline{y} \leqslant \overline{t} A(x,\overline{y}) \rightarrow \forall \overline{y} \leqslant \overline{t} A(x+1,\overline{y}))$ and

(b) $\neg\forall \overline{y} \leqslant \overline{t} A(a,\overline{y})$.

Note that atomic formulas are decidable in PV extended to intuitionistic logic and so in this theory $\neg\forall \overline{y} \leqslant \overline{t} A(a,\overline{y}) \equiv_i \neg\neg\exists \overline{y} \leqslant \overline{t} \neg A(a,\overline{y})$.

We want to prove the sentence $\forall x C(x)$ where $C(x)$ is the formula $\forall z \leqslant |a|(x+z=a \rightarrow \neg\neg\exists \overline{y} \leqslant \overline{t} \neg A(z,\overline{y}))$. First observe that $C(x)$ is equivalent in IPV to a doubly negated NP formula. For this it is enough to use the negative translation of Σ_1^b-replacement scheme which is provable in $PV + \neg\neg NP - IND$. The rest of the proof is similar to the former case.

(ii) A suitable version of the proof of (i) will work. To prove $PV+$ co$NP - LIND \vdash_i PV + \neg\neg NP - LIND$, in assumption (b) consider the sentence $\neg\exists \overline{y} \leqslant \overline{t} A(|a|,\overline{y})$ for some (term) a, and also consider the formula $\forall z \leqslant a(x+z=|a| \rightarrow \forall \overline{y} \leqslant \overline{t} \neg A(z, \overline{y}))$ as $B(x)$. To prove $PV + \neg\neg NP - LIND \vdash_i PV+$ co$NP - LIND$, make similar changes.

(iii) This is an immediate consequence of Proposition 2.2 and part (ii). □

Recall that the theory CPV is the classical closure of IPV and PV_1 is PV conservatively extended to first-order logic. It is known that, under the assumption $CPV = PV_1$, the polynomial hierarchy collapses, by a result of Krajicek, Pudlak and Takeuti (see [KPT]). Using the original construction, Buss, and independently Zambella, showed that if $CPV = PV_1$, then CPV proves a weaker form of the collapse (see [B5] and [Z]).

Theorem 2.4 If each of the following cases occurs, then $CPV = PV_1$:

(i) $IPV \vdash$ co$NP - PIND$.

(ii) $PV + NP - LIND \vdash_i$ co$NP - LIND$.

Proof We just prove case (i). The other is proved similarly.

(i) Assume $IPV \vdash$ co$NP - PIND$. Any ω-chain of (classical) models of $PV + NP - PIND$ ($\equiv CPV$) can be considered as a Kripke model of IPV whose underlying accessibility relation has order type ω (the proof is very similar to the one for Theorem 1.5. Also see [B2]). So, by a proof like the proof of Proposition 1.9, the union of the worlds in it should satisfy $PV+$ co$NP - IND$. Hence, this union should satisfy $PV + NP - PIND$ since $PV+$ co$NP - PIND \equiv_c PV + NP - PIND$. This shows that CPV is an inductive theory. Hence, using the well-known characterization of the inductive theories (see e.g. [CK, Th. 3.2.3]), CPV should be $\forall_2$. So, using $\forall_2$-conservativity of CPV over PV_1 (see [B1, Th. 5.3.6 and Coro. 6.4.8]), we get $CPV \equiv PV_1$. □

Note that, the notation $\equiv_c$ above, is used to denote equivalence in classical logic. We have to use this notation when we do not have specific names for theories at hand. The same is true about $\equiv_i$.

Note that the above proof actually shows that $IPV^+ \nvdash$ co$NP - PIND$ unless $CPV = PV_1$. The theory IPV^+ which was introduced by Buss [B2] apparently is stronger than IPV and is sound and complete with respect to CPV-normal Kripke structures.

Corollary 2.5 $IS_2^1 \nvdash \Pi_1^{b+} - PIND$, unless the Polynomial Hierarchy collapses.

Proof Use conservativity of IPV over IS_2^1 (see [CU, Theorem 2.4(i)]) and the above mentioned result in [KPT]. □

Acknowledgements Final version of this paper to appear in the Journal of Logic and Computation. I would like to thank two anonymous referees for suggestions that led to improvements in the presentation of the paper. This research was in part supported by a grant from IPM.

References

[A] J. Avigad, Interpreting Classical Theories in Constructive Ones, J. Symbolic Logic, 65 (2000) 1785-1812.

[B1] S. R. Buss, Bounded Arithmetic, Bibliopolis, 1986.

[B2] S. R. Buss, On Model Theory for Intuitionistic Bounded Arithmetic with Applications to Independence Results, in: Feasible mathematics, eds S. R. Buss and P. J. Scott, 1990, 27-47, Birkhauser.

[B3] S. R. Buss, A note on Bootstrapping Intuitionistic Bounded Arithmetic, Proof theory (Leeds, 1990), 149-169, Cambridge Univ. Press, Cambridge, 1992.

[B4] S. R. Buss and A. Ignjatovic, Unprovability of Consistency Statements in Fragments of Bounded Arithmetic, Ann. Pure Appl. Logic 74 (1995) 221–244.

[B5] S. R. Buss, Relating the Bounded Arithmetic and Polynomial Time Hierarchy, Ann. Pure Appl. Logic 75 (1995) 67-77.

[CK] C.C. Chang and J. Keisler, Model theory, North-Holland, 1990.

[CU] S. A. Cook and A. Urquhart, Functional Interpretations of Feasibly Constructive Arithmetic, Ann. Pure Appl. Logic, 63 (1993), 103-200.

[J1] J. Johannsen, A Model-Theoretic Property of Sharply Bounded Formulae, With Some Applications, Math. Logic Quart., 44 (1998), 205–215.

[J2] J. Johannsen, A Remark on Independence Results for Sharply Bounded Arithmetic, Math. Logic Quart., 44 (1998) 569-570.

[KPT] J. Krajicek, P. Pudlak and G. Takeuti, Bounded arithmetic and the polynomial hierarchy, International Symposium on Mathematical Logic and its Applications (Nagoya, 1988), Ann. Pure Appl. Logic 52 (1991) 143-153.

[T] G. Takeuti, Sharply Bounded Arithmetic and the Function $a \dot{-} 1$. Logic and computation (Pittsburgh, PA, 1987), 281–288, Contemp. Math., 106, Amer. Math. Soc., Providence, RI, 1990.

[TD] A. S. Troelstra and D. van Dalen, Constructivism in Mathematics, v.I, North-Holland, 1988. A.

[V] A. Visser, Submodels of Kripke Models, Arch. Math. Logic 40 (2001) 277-295.

[Z] D. Zambella, Notes on Polynomially Bounded Arithmetic, J. Symbolic Logic 61 (1996) 942-966.

Addresses:
Institute for Studies in Theoretical Physics and Mathematics (IPM), P.O. Box 19395-5746, Tehran, Iran; **AND**
Department of Mathematics, Shahid Beheshti University, Tehran, Iran.

Emails: ezmoniri@ipm.ir **and** ezmoniri@yahoo.com