Abelian Arboreal representations
FGC-HRI-IPM 2023

Carlo Pagano

Concordia University/MPIM (guest)

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- How quickly do arboreal degrees grow?
- When is an arboreal Galois group abelian?

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- It is a non-linear analogue of an l-adic representation.

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- This conjectured is inspired by Serre's open image theorem.
- *Large: (Size)* What about the actual size? How big is

$$\mathrm{Gal}(K(f^{-N}(\alpha))/K),$$

as N goes to ∞?

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Any of this is: **wide open** in general!

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- *Examples:* (x^d, ζ) or $(\pm T_d(x), \zeta + \zeta^{-1})$, $\zeta =$ a root of unity.
- **Conjecture, Andrews–Petsche, 2020:** For every number field these are the only abelian examples, up to conjugation.

Two questions

- How quickly arboreal degrees grow?
- **Expectation:** At least double-exponentially in the non-PCF case and at least exponentially in the PCF case.
- What are abelian arboreal Galois groups?
- **Expectation:** Only for pairs conjugate to (x^d, ζ) or $(\pm T_d(x), \zeta + \zeta^{-1})$.

Exponential lower bounds: PCF polynomials

Let K be a number field. We have the following.

Theorem 1, P., 2021

Assume GRH. Suppose that f is a PCF polynomials of degree $d \geq 2$. Let α be outside the critical orbits of f. Then there is $c(f, \alpha) > 0$ such that

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  Let us see how this works out exactly.

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- GRH gives a splitting prime of size about $\log(d_{K(f^{-N}(\alpha))})^{\frac{1}{2}-\epsilon}$. Hence this quantity grows exponentially in N.
- The discriminant is supported only at S and its log grows exponentially. The only possibility: degree grows exponentially!

Exponential lower bounds: unicritical polynomials

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Suppose that $f := x^d + c$ is *not* a PCF polynomials of degree $d \geq 2$. Then there is $c(f, \alpha) > 0$ such that

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- Apply the magic modulo a suitably chosen prime.

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- Now iterate!

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- The latter follows exploiting the magic of PCF that are critically periodic.
- The magic will come back!

Progress on Andrews–Petsche: reduction to the PCF case

Theorem 3, Ferraguti–P., 2023

If a unicritical polynomial $x^d + c$ over any number field K, gives abelian arboreal Galois group for some α, then the orbit of 0 is preperiodic.

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- If the orbit were infinite one would get curves of very high genus having infinitely many rational points.

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The coordinate projections ϕ_i are basically $f^i(0) - \alpha$ modulo squares. This gives you the curves!

Intermezzo: a rigidity theorem

- This principle played a key role in a previous work.

Theorem, Casazza–Ferraguti–P, 2019

The list of maximal subgroups of $\Omega_\infty(2)$ along with $\Omega_\infty(2)$ consists of pairwise distinct isomorphism classes of profinite groups.

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- For all but finitely many i, the largest number of connected components in the *graphs of commutativity* of $\Omega_\infty(2)^{(i-\mathrm{Fr.})}$ is equal to 1 iff $\underline{a} = 0$ and otherwise equals 2^{N+1}, where N is the largest non-zero coordinate.

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- This is essentially a consequence of the unidimensionality principle!
- It reconstructs the largest 1. The previous 1's are detected by looking which terms of the series $\Omega_\infty(2)^{i-\mathrm{Fr.}}$ are topologically generated by involutions.

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- The largest number of connected components is considered among the set of generators not containing the identity.
- We are currently generalizing to p odd: it turns out one iterates the $(p-1)$-th piece of the lower central series!
- For general p one has that isomorphic groups occur iff the vectors have same support, which happens iff the two subgroups are $\mathrm{Aut}_{\mathrm{top.gr.}}(\Omega_\infty(p))$-conjugate.

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- In the result above we have achieved exactly this reduction for unicritical polynomials.

Recap

- Andrews–Petsche conjectured that only $(x^d, \zeta), (\pm T_d(x), \zeta + \zeta^{-1})$ yield abelian Galois groups (up to conjugation).
- Jones' conjecture predicts at least that one should be able to restrict to the PCF-case.
- In the result above we have achieved exactly this reduction for unicritical polynomials.

  So we can now focus entirely on the PCF case.

Progress on Andrews–Petsche: the periodic case

Among the PCF we settle all of the periodic ones:

Theorem 4, Ferraguti–P., 2023

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- From there one shows that f preserves the unit circle.
- But $x^d + c$ preserves the unit circle only when $c = 0$!

Progress on Andrews–Petsche: $\mathbb{Q}$ and quadratic number fields

We have the following:

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Andrews–Petsche conjecture holds for all monic unicritical polynomial over $\mathbb{Q}$ and over quadratic number fields.

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- For $\mathbb{Q}$ and for all quadratic polynomials (Ferraguti–P. (2020), using the unidimensionality principle and local class field theory).
- For more general rational functions over $\mathbb{Q}$ (Ferraguti–Ostafe–Zannier, 2022). More on this later.

$\mathbb{Q}$ and quadratic number fields: ideas

- The list of PCF polynomials to look at is

$$\{x^d, x^2 - 2, x^{2d} - 1, x^{4d+3} \pm i, x^{6d+4} \pm \zeta_6, x^{6d} + \zeta_3, x^2 \pm i\}.$$

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The cases $\{x^{6d} + \zeta_3, x^2 \pm i\}$

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  We use a method of Amoroso–Zannier (to lower bound heights in abelian extensions) to reduce the range to $d \leq 36$. Not directly their estimate. The remaining cases are done with Magma.

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For all d there exists a finite set $U_d \subseteq \mathbb{Q}^{\text{sep}}$ such that for all number fields K and all u in K and not in U_d, there are only finitely many α in K such that $(u \cdot x^d + 1, \alpha)$ gives abelian image.

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The reduction "abelian implies PCF": we know it for every polynomial over any number field and not only for unicriticals (Ferraguti–Ostafe–Zannier, 2022).

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- So: any source of super-exponential lower bounds would directly rule out polynomials!
- Conversely the only currently known cases with an exponential growth are precisely Chebichev and power polynomials.

Thanks for the attention!