Applications of the Combinatorial Nullstellensatz

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Plan

The Combinatorial Nullstellensatz and the polynomial method

Applications in various fields

Additive results on sequences

Sequences or geometry in finite fields

Addition Theorems in $\mathbb{F}_p$

Plan

The Combinatorial Nullstellensatz

Theorem (Alon 1999)

K a field and P a polynomial from $K[X_1, \ldots, X_d]$.
If $\deg(P) = \sum_{i=1}^{d} k_i$ and its coefficient of $\prod_{i=1}^{d} X_i^{k_i}$ is not zero,

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If $\deg(P) = \sum_{i=1}^{d} k_i$ and its coefficient of $\prod_{i=1}^{d} X_i^{k_i}$ is not zero,
Then whatever $A_1, \ldots, A_d$ subsets of $\mathbb{K}$ satisfying $|A_i| > k_i$, there is $(a_1, \ldots, a_d) \in A_1 \times \cdots \times A_d$ such that :

$$P(a_1, \ldots, a_d) \neq 0.$$

Another formulation

Theorem

K a field and P a polynomial from $K[X_1, \ldots, X_d]$. Let $A_1, \ldots, A_d$ a family of subsets of K.
Denoting $g_i(X_i) = \prod_{a_i \in A_i}(X_i - a_i)$.

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Denoting $g_i(X_i) = \prod_{a_i \in A_i}(X_i - a_i)$.
If P vanishes on $A_1 \times \cdots \times A_d$, then there are polynomials $h_i \in K[X_1, \ldots, X_d]$, with $\deg(h_i) \leqslant \deg(P) - deg(g_i)$ such that:

$$P(\underline{X}) = \sum_{i=1}^{d} h_i(\underline{X}) g_i(X_i).$$

The coefficient formula

Theorem (Schauz, Karasev-Petrov, Lasoń)

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If $\deg(P) = \sum_{i=1}^{d} k_i$ then whatever the family of subsets $A_1, \ldots, A_d$, such that $|A_i| = k_i + 1$, the coefficient of $\prod_{i=1}^{d} X_i^{k_i}$ in P is:

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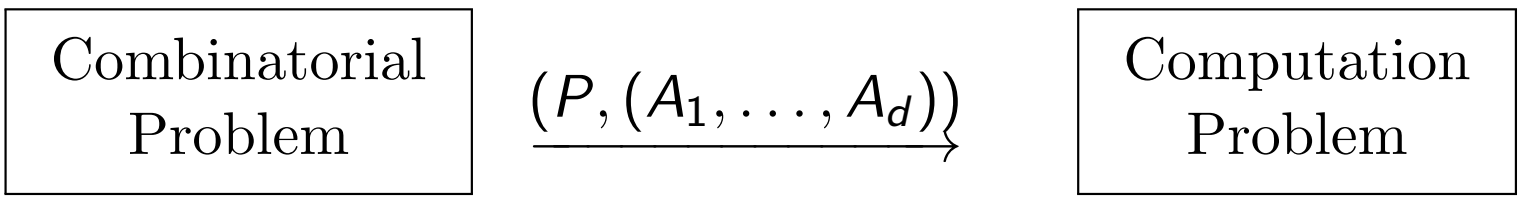
$$c = \sum_{\underline{a} \in \prod A_i} \frac{P(\underline{a})}{\prod g_i'(a_i)}.$$

The polynomial method



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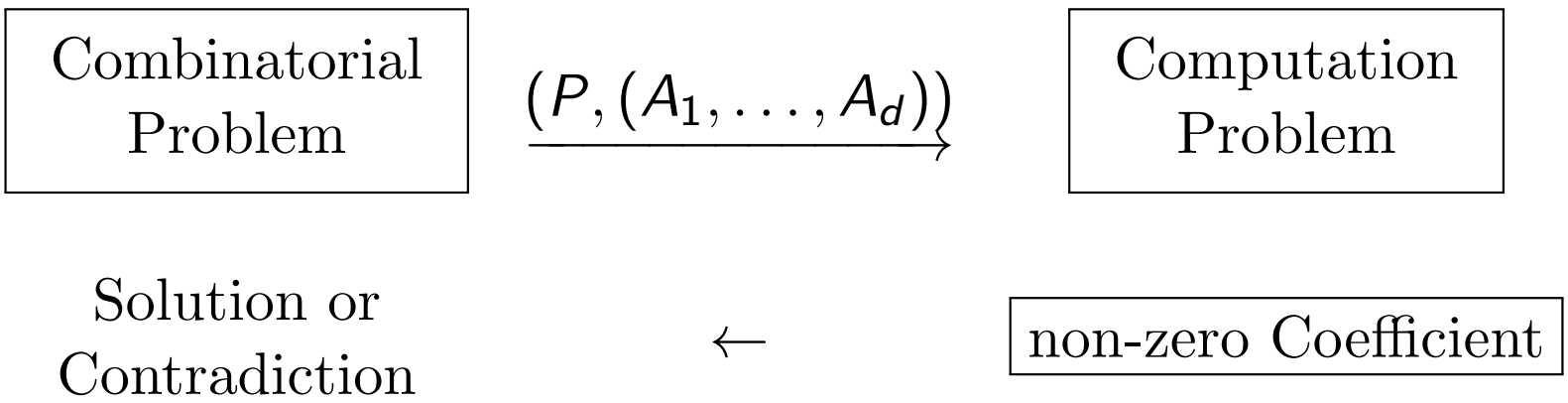
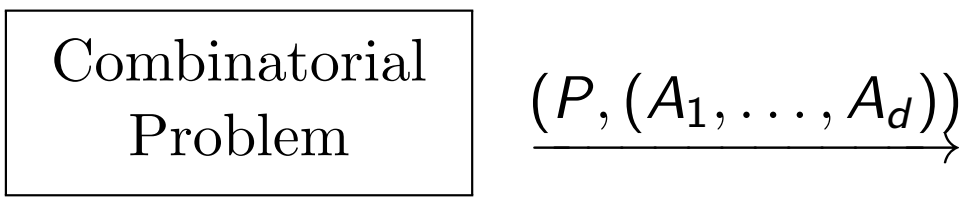


The polynomial method revisited



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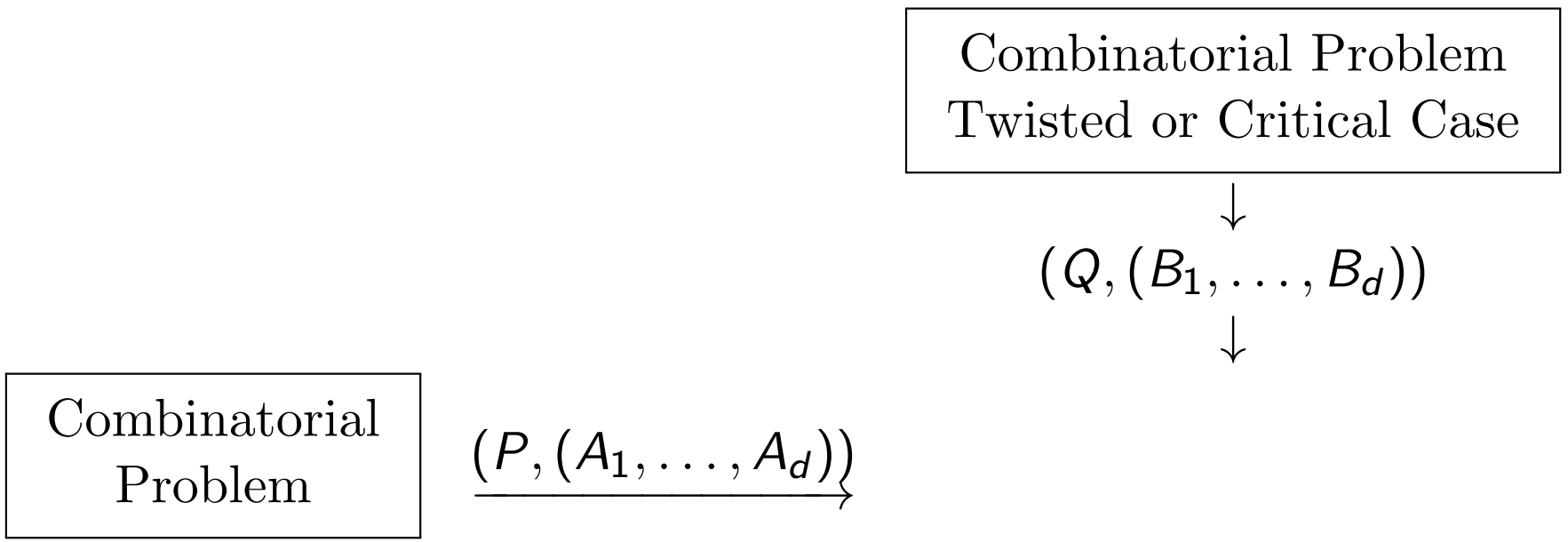

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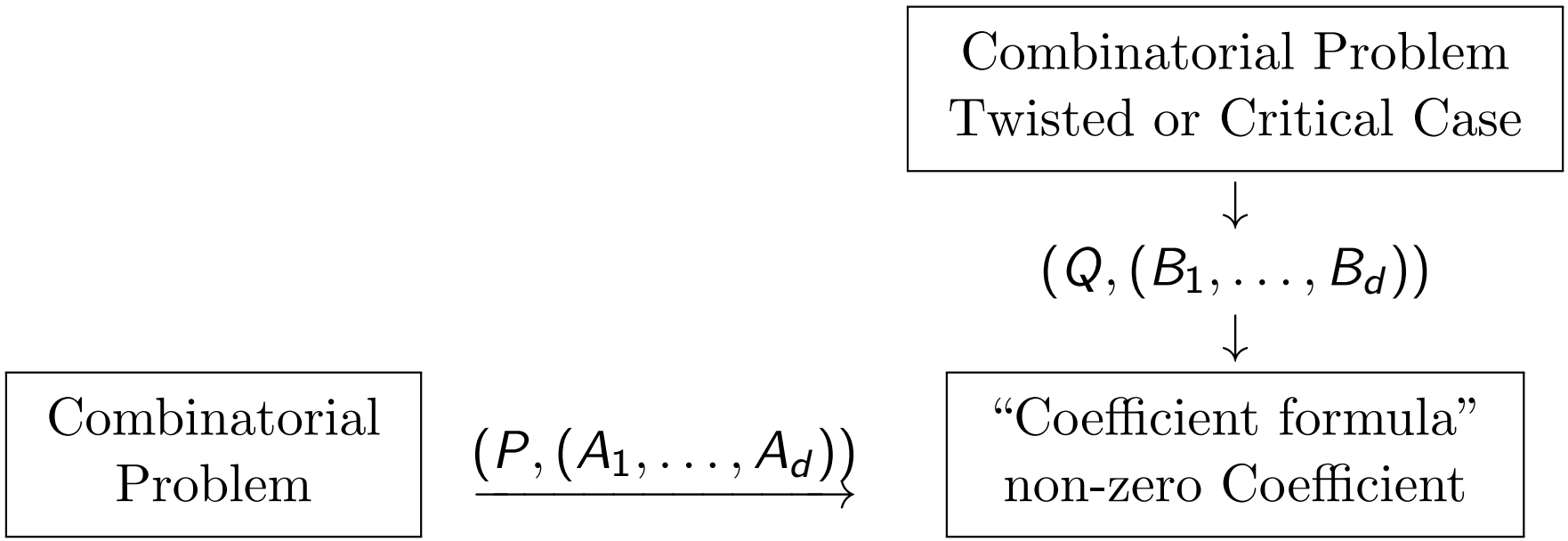

Plan

Theorem (Alon, Friedland, Kalai)

$p \in \mathbb{P}$, $G = (V, E)$ simple graph with average degre $\bar{d} > 2p - 2$ and max degre $\Delta \leqslant 2p - 1$, then G admits a p-regular subgraph.

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In $\mathbb{F}_p$, $\forall e \in E$, $A_e = \{0, 1\}$

$$P((X_e)_{e\in E}) = \prod_{v\in V}\left(1 - \left(\sum_{e\in E} a_{ve}X_e\right)^{p-1}\right) - \prod_{e\in E}(1 - X_e).$$

Theorem (Komjáth's conjecture, Alon-Furedi)

In $\mathbb{R}^d$, *to cover* $\{0,1\}^d \setminus (0,\ldots,0)$ *by hyperplanes not containing* $(0,\ldots,0)$, *one needs at least* d *hyperplanes.*

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Given $m \leq d$ hyperplanes, $A_i = \{0,1\}$:

$$P(X_1,\ldots,X_d) = \prod_{i=1}^{m} \underbrace{\left(\sum_{j=1}^{d} a_{ij}X_j - 1\right)}_{\text{normalised equation}} - (-1)^{m-d} \prod_{j=1}^{d} (X_j - 1).$$

Theorem (Chevalley-Warning)

In $\mathbb{F}_p$*, whenever* $\sum \deg(P_i) < d$ *the number of solutions to*

$$\begin{cases} P_i(X_1, \ldots, X_d) = 0 \end{cases}$$

is a multiple of p*.*

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is a multiple of p.

$\prod_{i=1}^{d} A_i = \mathbb{F}_p^d$,

$$\prod \left(1 - P_i^{p-1}(X_1, \ldots, X_d)\right) - \sum_{\underline{s} \in S} L_{\underline{s}}(X_1, \ldots, X_d).$$

Theorem (Conjecture de Dyson, Karasev-Petrov)

Given integers $a_1, \ldots, a_n$, *the fraction*

$$\prod_{1 \leq i \neq j \leq n} \left(1 - \frac{X_i}{X_j}\right)^{a_i}$$

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Coefficient of $\prod_{i=1}^{n} X_i^{a-a_i}$ in $\prod_{1 \leq i < j \leq n} (-1)^{a_j} (X_j - X_i)^{a_j + a_i}$.
Consider the sets $A_i = [0, (\Sigma a_j) - a_i]$ and

$$\prod_{1 \leq i < j \leq n} \prod_{s=-a_i+1}^{a_j} (X_j - X_i + s).$$

Plan

The permanent Lemma

Theorem (Alon)

K a field, A an $n \times n$ matrix with non zero permanent, $b \in K^n$, and $S_i \subset \mathbb{K}$, $i = 1..n$, $|S_i| = 2$. There exists $s = (s_1, \ldots, s_n) \in S_1 \times \cdots \times S_n$, such that As and b are coordinatewise distincts.

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$\prod_{i=1}^{n} A_i = \prod_{i=1}^{n} S_i$,

$$\prod_{i=1}^{n} \left(\sum_{j=1}^{n} a_{i,j} X_j - b_i \right),$$

coefficient of $\prod_{i=1}^{n} X_i$ is $Per(A) \neq 0$.

Theorem (Erdős, Ginzburg, Ziv)

G abelian finite group, $|G| = n$. Whathever $(g_1, g_2, \ldots, g_{2n-1})$ elements of G. There exists a zerosum subsequence of length n.

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$\prod_{i=1}^{p-1} A_i = \prod_{i=1}^{p-1} \{g_i, g_{i+p-1}\}$

$$A = \underbrace{\begin{pmatrix} 1 & \ldots & 1 \\ \vdots & & \vdots \\ 1 & \ldots & 1 \end{pmatrix}}_{p-1},$$

$$\underline{b} = (-g_{2p-1} + 1, \ldots, -g_{2p-1} + (p-1)).$$

Snevily's conjecture

G a finite abelian group of odd order.

$a_1, \ldots, a_k$, distinct elements	There is π, such that
$b_1, \ldots, b_k$, distinct elements	$a_1 + b_{\pi(1)}, \ldots, a_k + b_{\pi(k)}$
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$G = \mathbb{Z}/n\mathbb{Z}$ (Dasgupta, Karolyi, Serra, Szegedy),
$\prod_{i=1}^{k} A_i = \{g^{a_i} \mid i = 1..k\}^k \subset \mathbb{F}_{2^d}^k$,

$$P(X_1, \ldots, X_k) = \prod_{1 \leqslant j < i \leqslant k} (X_i - X_j)\,(\alpha_i X_i - \alpha_j X_j)\,,$$

with $\alpha_i = g^{b_i}$.

Theorem (Kemnitz conjecture - Rónyai)

In a sequence of $4p-2$ *elements* $((a_i, b_i))$ *of* $\mathbb{F}_p^2$, *there is a* 0*-sum subsequence of length* p*:*

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$\prod_{i=1}^{4p-2} A_i = \{0,1\}^{4p-2}$,

$$\left(1-\left(\sum_{i=1}^{4p-2} a_i X_i\right)^{p-1}\right)\left(1-\left(\sum_{i=1}^{4p-2} b_i X_i\right)^{p-1}\right)$$

$$\times\left(1-\left(\sum_{i=1}^{4p-2} X_i\right)^{p-1}\right)\left(\sum_{\substack{I\in[1,4p-2]\\|I|=p}}\prod_{i\in I} X_i - 2\right) + (2L_{\underline{0}}(\underline{X})).$$

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Problem "à la Vinatier"

Theorem (Gács, Héger, Nagy, Pálvögyi)

In $\mathbb{F}_q^n$, $n \leqslant q$, $H_{i,j} = \{\underline{X} | X_i = X_j\}$, *whenever* $H \subset \bigcup_{i \neq j} H_{i,j}$.

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- $H = H_{i,j}$,
- $n = q$, $H = \{\underline{X} | \alpha(X_i - X_j) + \sum X_k = 0\}$,
- $n = q - 1$, $H = \{\underline{X} | X_j + \sum X_k = 0\}$.

Theorem

$(a_1, \ldots, a_q) \in \mathbb{F}_q^q$

There is no pairwise distinct $(b_1, \ldots, b_q) \in \mathbb{F}_q^q$ *such that* $\sum_{i=1}^{q} a_i b_i = 0$.

$\iff$ *There are* $(a, b) \in \mathbb{F}_q$, $b \neq 0$ *such that* $a_i = a + b$, $a_j = a - b$, *and* $k \neq i, j$, $a_k = a$.

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$\prod_{i=1}^q A_i = \mathbb{F}_q^q$,

$$G(\underline{Y}) = \left(\left(\sum_{i=1}^{k} Y_i \right)^{q-1} - 1 \right) \begin{vmatrix} a_1^{k-1} & a_1^{k-1} Y_1 & \ldots & Y_1^{k-1} \\ a_2^{k-1} & a_2^{k-1} Y_2 & \ldots & Y_2^{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_k^{k-1} & a_k^{k-1} Y_k & \ldots & Y_k^{k-1} \end{vmatrix}.$$

A question of Erdős

$A = (a_1, \ldots, a_\ell)$ a sequence of $\mathbb{F}_p^\times$.
$\mathcal{S}_A$: set of $(0-1)$-solutions of

$$a_1 x_1 + \cdots + a_\ell x_\ell = 0.$$

$$\mathcal{S}_A = A^\perp \cap \{0,1\}^\ell$$

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Set

$$\dim(A) = \dim(< \mathcal{S}_A >).$$

Theorem (B.-Girard)

$A = (a_1, \ldots, a_\ell)$ a sequence of $\ell > p$ elements of $\mathbb{F}_p^\times$:

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Theorem (B.-Girard)

$A = (a_1, \ldots, a_p)$ a sequence of p elements of $\mathbb{F}_p^\times$:

- $\dim(A) = 1$,
$$(a_1, \ldots, a_p) = (r, \ldots, r).$$
- $\dim(A) = p - 2$, $\exists t \in [1, p-3]$,
$$(a_{\sigma(1)}, \ldots, a_{\sigma(p)}) = (\underbrace{r, \ldots, r}_{t}, \underbrace{-r, \ldots, -r}_{p-2-t}, -(t+1)r, -(t+1)r).$$
- $\dim(A) = p - 1$.

$\mathcal{S}_A \subset \mathcal{S}_B$,

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$$\prod A_i = \prod \Sigma_i,$$

$$P(X_1, \ldots, X_d) = \left(\underbrace{\sum_{i=1}^{d} \lambda_i X_i}_{\sum_{i\in I} b_i} \right) \left(\left(\underbrace{\sum_{i=1}^{d} X_i}_{\sum_{i\in I} a_i} \right)^{p-1} - 1 \right).$$

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Theorem (Cauchy-Davenport)

p a prime number, A and B two subsets of $\mathbb{F}_p$, then:

$$|A+B| \geqslant \min\{p, |A|+|B|-1\}.$$

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$(p > (|A|-1)+|B-1|)$
$A \times B$,

$$\prod_{c \in A+B} (X+Y-c)$$

The coefficient of $X^{|A|-1}Y^{|B|-1}$ is $\binom{(|A|-1)+(|B|-1)}{|A|-1} \neq 0$.

Conjecture (Erdős-Heilbronn)

p a prime number, $A \subset \mathbb{F}_p$, then:

$$|\{a_1 + a_2 \mid a_i \in A,\ a_1 \neq a_2\}| \geqslant \min\{p, 2|A| - 3\}$$

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Define:

$$h^{\wedge}A = \{a_1 + \cdots + a_h \mid a_i \in A,\ a_i \neq a_j\}$$

Theorem (Dias da Silva, Hamidoune)

Let p be a prime number and $A \subset \mathbb{F}_p$. Let $h \in [1, |A|]$, one has,

$$|h^{\wedge}A| \geqslant \min\{p, 1 + h(|A| - h)\}.$$

$$(p > h(|A| - h))$$

$$\left|\begin{array}{lll} A_1 & = \left\{a_1, \ldots, a_{|A|-h}, a_{|A|-h+1}\right\} & \\ \vdots & \vdots & \ddots \\ A_{h-1} & = \left\{a_1, \quad \cdots \quad , a_{|A|-1}\right\} & \\ A_h & = \left\{a_1, \quad \cdots \quad , a_{|A|-1}, a_{|A|}\right\}, & \end{array}\right.$$

$$\prod_{c \in h^{\wedge} A} (X_1 + \cdots + X_h - c) \left(\prod_{1 \leqslant i < j \leqslant h} (X_j - X_i) \right).$$

Let A be a subset of $\mathbb{F}_p$, define

$$\Sigma^*(A) = \left\{ \sum_{x \in I} x \mid \emptyset \subsetneq I \subset A \right\}$$

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Theorem (B.)

p a odd prime number, $A \subset \mathbb{F}_p$, such that $A \cap (-A) = \emptyset$. One has

$$|\Sigma(A)| \geqslant \min \left\{ p, 1 + \frac{|A|(|A|+1)}{2} \right\},$$

$$|\Sigma^*(A)| \geqslant \min \left\{ p, \frac{|A|(|A|+1)}{2} \right\}.$$

$$(p > \tfrac{d(d+1)}{2} + 1)$$

$$A = \{2a_1, \ldots, 2a_d\},\ \Sigma(A) = \left(\sum_{i=1}^{d} a_i\right) + \sum_{i=1}^{d} \{-a_i, a_i\}$$

$$\left|\begin{array}{ll} A_1 & = \{a_1, \ldots, a_d, -a_1\} \\ \vdots & \qquad\qquad\qquad\quad \ddots \\ A_d & = \{a_1, \ldots, a_d, -a_1, \ldots, -a_d\}, \end{array}\right.$$

$$\prod_{c \in \Sigma(A)} (X_1 + \cdots + X_d - c) \left(\prod_{1 \leqslant i < j \leqslant d} (X_j^2 - X_i^2) \right)$$

Define

$$\Sigma_\alpha(A) = \{a_1 + \cdots + a_k \mid a_i \in A,\ \alpha \leq k \leq |A|,\ a_i \neq a_j\}$$
$$\Sigma^\alpha(A) = \{a_1 + \cdots + a_k \mid a_i \in A,\ 0 \leq k \leq |A| - \alpha,\ a_i \neq a_j\}.$$

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Theorem (B.)

$A \subset \mathbb{F}_p$ such that $A \cap (-A) = \emptyset$, then:

$$|\Sigma_\alpha(A)| = |\Sigma^\alpha(A)| \ge \min\left\{p, \frac{|A|(|A|+1)}{2} - \frac{\alpha(\alpha+1)}{2} + 1\right\}$$

Set $A = \{2a_1, 2a_2, \ldots, 2a_d\}$ et $m = \sum_{i=1}^{d} a_i$.
Let C such that $\Sigma^{\alpha}(A) \subset C$ and
$|C| = \min\left\{p-1, \frac{d(d+1)}{2} - \frac{\alpha(\alpha+1)}{2}\right\}$

$$P_{d,\alpha}(\underline{X}) = \prod_{x \in C}(X_1+\cdots+X_d+m-x) \prod_{1 \le i < j \le d}(X_j - X_i) \prod_{\substack{1 \le i < j \le d \\ \text{and } j > \alpha}}(X_j + X_i)$$

$$
\begin{array}{lcl}
A_1 & = & \{-a_d, \ldots, -a_\alpha\} \\
\vdots & & \vdots \qquad\qquad \ddots \\
A_\alpha & = & \{-a_d, \quad \ldots \quad , -a_1\} \\
A_{\alpha+1} & = & \{-a_d, \quad \ldots \quad , -a_1, \quad a_1, \ldots, a_{\alpha+1}\} \\
\vdots & & \vdots \qquad\qquad\qquad \vdots \quad \vdots \qquad\qquad \ddots \\
A_d & = & \{-a_d, \quad \ldots \quad , -a_1, \quad a_1, \quad \ldots \quad , a_d\}
\end{array}
$$

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\begin{array}{lcl}
B_1 & = & \{-d, \ldots, -\alpha\} \\
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B_{\alpha+1} & = & \{-d, \quad \ldots \quad , -1, \quad 1, \ldots, \alpha+1\} \\
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B_d & = & \{-d, \quad \ldots \quad , -1, \quad 1, \qquad \ldots \qquad , d\}
\end{array}
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