Fair partition of a convex planar pie

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joint work with Arseniy Akopyan [2] and Sergey Avvakumov [2]

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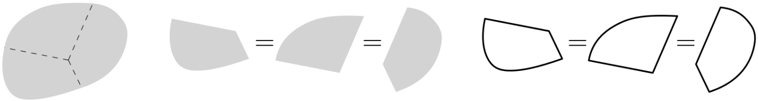
The problem statement

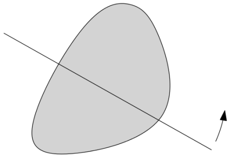
Question (Nandakumar and Ramana Rao, 2008)

Given a positive integer m and a convex body K in the plane, can we cut K into m convex pieces of equal areas and perimeters?



Previously known results

- For $m = 2$ it can be done by a simple continuity argument.



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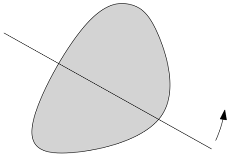

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- A generalization of the continuity argument through an appropriate Borsuk–Ulam-type theorem yields a proof for $m = p^k$ with p prime. The topological tool was used previously by Viktor Vassiliev for a different problem (1989). The fair partition result for $m = 2^k$ was proved explicitly by Mikhail Gromov (2003).

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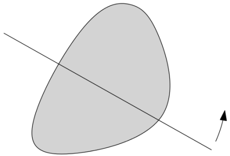

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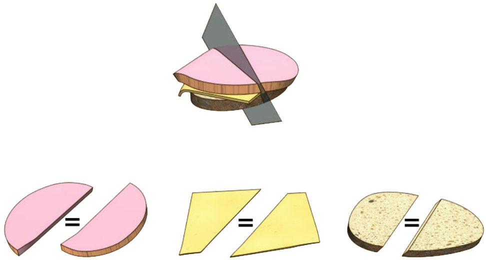
- For m, which is not a prime power, this direct technique fails.

A classical example: the ham sandwich theorem

Theorem

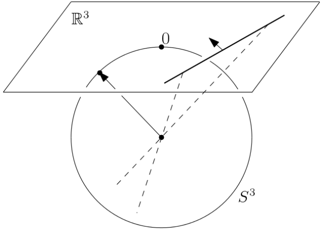
Any 3 sufficiently nice probability measures in $\mathbb{R}^3$ can be simultaneously equipartitioned by a plane.



https://curiosamathematica.tumblr.com

Scheme of proof

- The configuration space is the sphere S^3, which naturally



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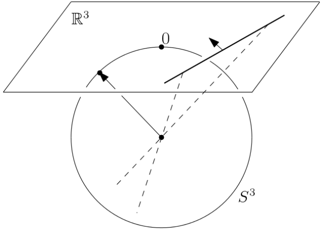

- The test map $f: S^3 \to \mathbb{R}^3$ sends an oriented plane $u \in S^3$ to the point $f(u) \in \mathbb{R}^3$ whose i-th coordinate is the difference of the values of the i-th measure on the two corresponding halfspaces.

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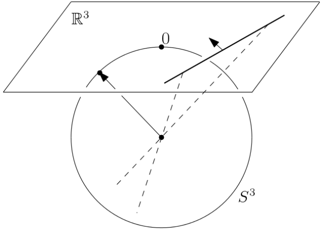

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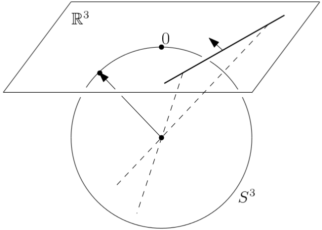

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- Solutions are in $f^{-1}(0)$.
- This map is $\mathbb{Z}_2$-equivariant, i.e., $f(-u) = -f(u)$, and the classical Borsuk–Ulam theorem guarantees that any such map must have a zero, which yields the desired equipartition. $\square$

Convex fair partitions for prime power

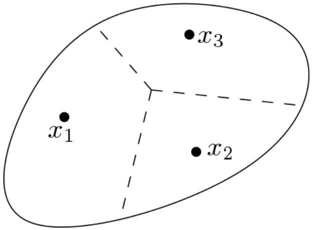
Theorem (Karasev, Hubard, Aronov, Blagojević, Ziegler, 2014)

If m is a power of a prime then any convex body K in the plane can be partitioned into m parts of equal area and perimeter.

The case $m = 3$ was done first by Bárány, Blagojević, and Szűcs. In dimension $n \geq 3$ a similar result with equal volumes and equal $n - 1$ other continuous functions of m convex parts was also established for $m = p^k$.

Configuration space

$F(m)$ is the space of m-tuples of pairwise distinct points in $\mathbb{R}^2$. Given $\bar{x} \in F(m)$ we can use Kantorovich theorem on optimal transportation to equipartition K into m parts of equal area. The partition is a weighted Voronoi diagram with centers in $\bar{x}$.



$\bar{x} \in F(3)$.

No need to consider partitions not in $F(m)$

The space $F(m)$ is smaller than the space $E(m)$ of all equal are convex partitions. However, there is an $\mathfrak{S}_m$-equivariant map

$$F(m) \to E(m),$$

given by the Kantorovich theorem, and an $\mathfrak{S}_m$-equivariant map

$$E(m) \to F(m),$$

sending a partition into its centers of mass. The maps do not commute, but show that the spaces are equivalent from the points of view of plugging them into a Borsuk–Ulam-type theorems.

Further simplification of $F(m)$

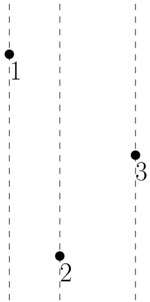
The dimension of $F(m)$ is $2m$. We can further simplify it.

Lemma (Blagojević and Ziegler, 2014)

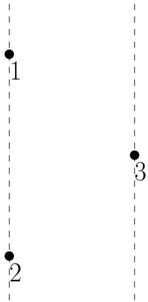
Space $F(m)$ retracts $\mathfrak{S}_m$-equivariantly to its subpolyhedron $P(m) \subset F(m)$ with $\dim(P(m)) = m - 1$.

This lemma allows to imagine how the solution changes if we consider a family of problems depending on a parameter.

Cellular decomposition of $F(3)$



A 6-cell.



A 5-cell.

A 4-cell.

Equivariant map

Let the map $f : P(m) \to \mathbb{R}^m$ send a generalized Voronoi equal area partition into the *perimeters* of the m parts. The test map is $\mathfrak{S}_m$-*equivariant*, if $\mathfrak{S}_m$ acts on $\mathbb{R}^m$ by permutations of the coordinates.

A partition corresponding to $u \in P(m)$ solves the problem if $f(u) \in \Delta := \{(x, x, \ldots, x) \in \mathbb{R}^m\}$.

Homology of the solution set

Theorem (Blagojević and Ziegler)

If $m = p^k$ is a prime power and $f : P(m) \to \mathbb{R}^m$ is an $\mathfrak{S}_m$-equivariant map in general position, then $f^{-1}(\Delta)$ is a non-trivial 0-dimensional cycle modulo p in homology with certain twisted coefficients.

If m is not a prime power then there exists an $\mathfrak{S}_m$-equivariant map $f : P(m) \to \mathbb{R}^m$ with $f^{-1}(\Delta) = \emptyset$.

Our main result

Theorem (Akopyan, Avvakumov, K.)

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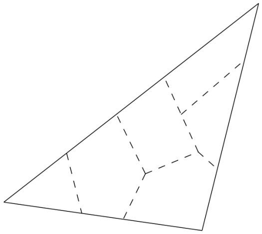
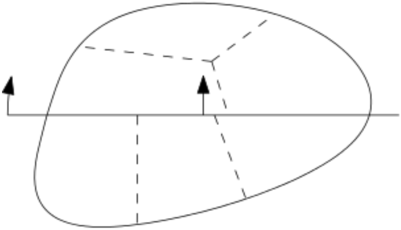
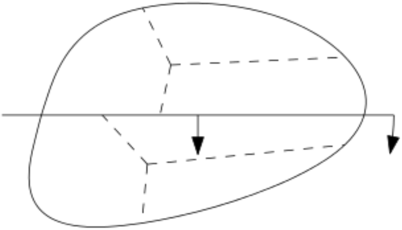
When m is not a prime power, the theorem was unknown even for simplest K, e.g., for generic triangles.



"Naive" continuity argument

- "Naive" argument for $m = 6$ (the smallest non-prime-power):





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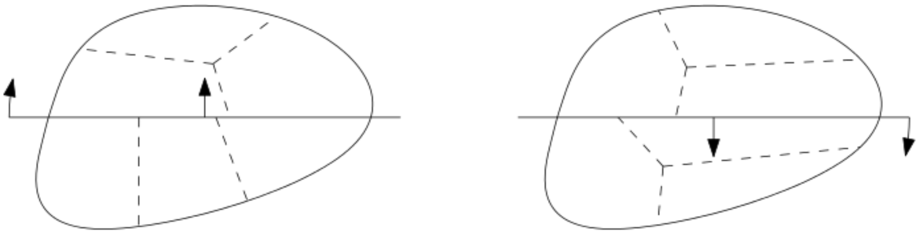

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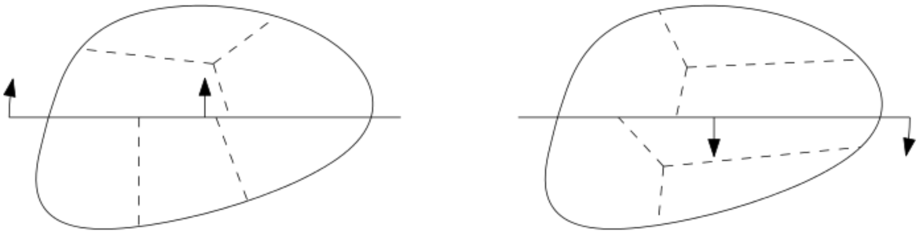

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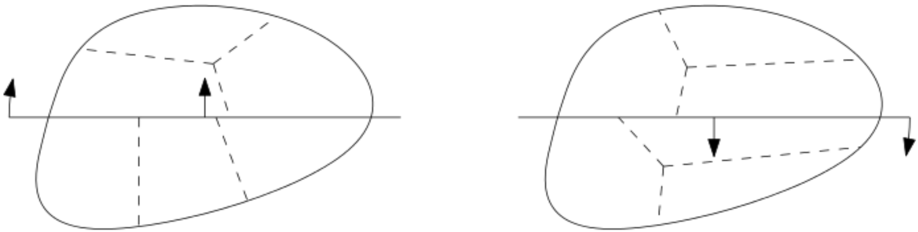

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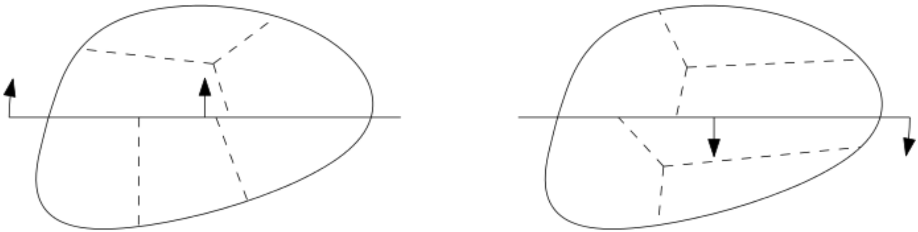
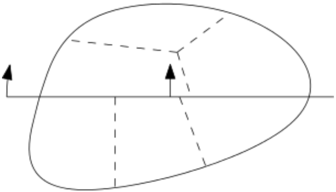
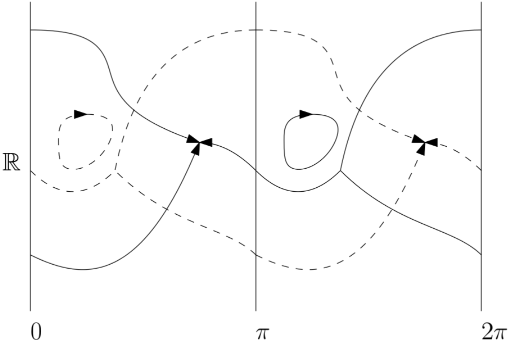
  - Pick a direction and a halving line in that direction.
  - Fair partition each half into 3 pieces.
  - Rotate the direction.
- There are difficulties arguing this way, because the partitions in three parts may not depend continuously on parameters of the half subproblem.



Proof sketch for $m = 6$

As we rotate the direction, plot the *perimeters* of *all* the solutions, the language of multivalued functions must be useful.

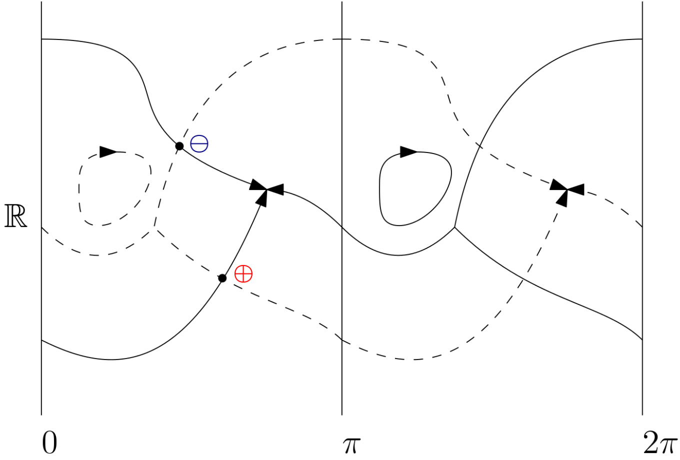




Solid and dashed are perimeters on the left and right, resp. Solid/dashed intersections are fair partitions.

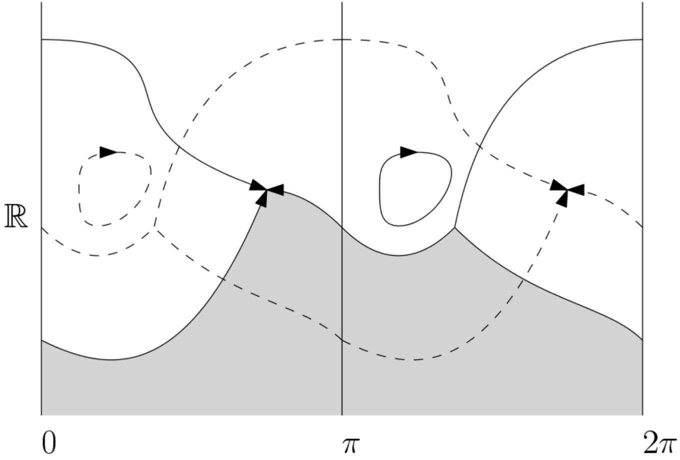
Number of solutions

In this particular example the number of solutions, with signs, is 0!



Proof sketch for $m = 6$

Solid graph separates the bottom from the top, from the homology modulo 3 description of the solution set by Blagojević and Ziegler.



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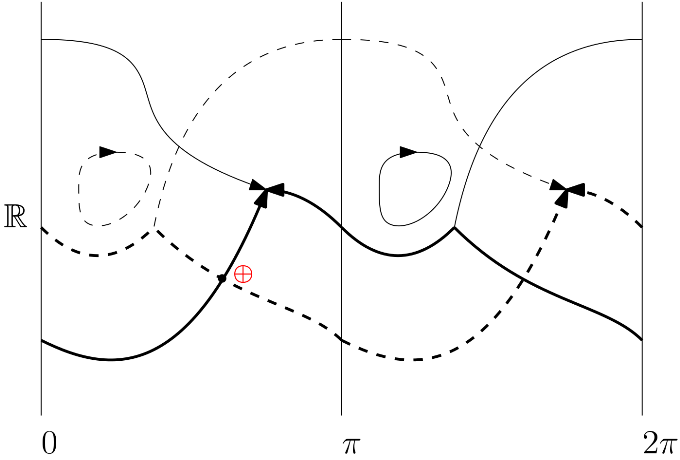
After choosing an appropriate subgraph of the multivalued function, bold solid and bold dashed curves intersect at 1 point, modulo 2. □



Plan of the proof for arbitrary m

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- Step $i \to i - 1$ equializing the perimeter values in parts of $(i - 1)$th stage region, keeping the separation property for the new multivalued function, the common value of the perimeter.

Summary

Generalizations:

- "Area" can be any finite Borel measure, zero on hyperplanes.
- "Perimeter" can be any Hausdorff-continuous function on convex bodies (e.g., diameter).
- Unknown, if we replace "area" with an arbitrary (i.e., non-additive) rigid-motion-invariant continuous function of convex bodies.
- If we want to equalize the volumes and two perimiter-like functions in $\mathbb{R}^3$, then it is possible for $m = p^k$ (K., Aronov, Hubard, Blagojević, Ziegler), but our current method does not work already for $m = 2p^k$.

Full version is arXiv:1804.03057.

Summary

Thank you for your attention!

Full version is arXiv:1804.03057.