Thresholds in random graphs with focus on thresholds for k-regular subgraphs

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Binomial random graph $\mathcal{G}(n, p)$

Let $0 \leq p \leq 1$ (usually $p = p(n) \to 0$ as $n \to \infty$).

Start with an empty graph with vertex set $[n] := \{1, 2, \ldots, n\}$.

Perform $\binom{n}{2}$ Bernoulli experiments inserting edges independently with probability p.

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Perform $\binom{n}{2}$ Bernoulli experiments inserting edges independently with probability p.

Alternatively, for $0 \leq m \leq \binom{n}{2}$, assign to each graph G with vertex set $[n]$ and m edges a probability

$$\mathbb{P}(G) = p^m (1-p)^{\binom{n}{2}-m}.$$

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Model introduced by Gilbert (1959) and popularized in the seminal papers of Erdős and Rényi (1959, 1960).

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Perform $\binom{n}{2}$ Bernoulli experiments inserting edges independently with probability p.

The results are asymptotic in nature ($n \to \infty$).

We say that a given event holds asymptotically almost surely (a.a.s.) if the probability it holds tends to 1 as $n \to \infty$.

Thresholds and Sharp Thresholds

One of the most striking behaviour of random graphs is the appearance and disappearance of certain graph properties.

A function $p^* = p^*(n)$ is a threshold for a monotone increasing property $\mathcal{P}$ in the random graph $\mathcal{G}(n,p)$ if

$$\lim_{n\to\infty} \mathbb{P}(\mathcal{G}(n,p) \in \mathcal{P}) = \begin{cases} 0 & \text{if } p/p^* \to 0 \\ 1 & \text{if } p/p^* \to \infty. \end{cases}$$

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(Note that the thresholds defined above are not unique.)

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Alternatively, one can say that:
– if $p \ll p^*$, then a.a.s. $\mathcal{G}(n,p) \notin \mathcal{P}$
– if $p \gg p^*$, then a.a.s. $\mathcal{G}(n,p) \in \mathcal{P}$

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Theorem (Bollobás and Thomason, 1986)

Every non-trivial monotone graph property has a threshold in the random graph $\mathcal{G}(n,p)$.

Thresholds and Sharp Thresholds

A function $p^* = p^*(n)$ is a sharp threshold for a monotone increasing property $\mathcal{P}$ in the random graph $\mathcal{G}(n,p)$ if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}(\mathcal{G}(n,p) \in \mathcal{P}) = \begin{cases} 0 & \text{if } p/p^* \leq 1-\varepsilon \\ 1 & \text{if } p/p^* \geq 1+\varepsilon. \end{cases}$$

Connectivity

Theorem (Erdös and Rényi, 1959)

Let $p = p(n) = \frac{\log n + c_n}{n}$. *Then,*

$$\lim_{n\to\infty} \mathbb{P}(\mathcal{G}(n,p) \textit{ is connected}) = \begin{cases} 0 & \textit{if } c_n \to -\infty \\ e^{-e^{-c}} & \textit{if } c_n \to c \\ 1 & \textit{if } c_n \to \infty. \end{cases}$$

Sharp threshold: $p^* = \log n / n$.

Connectivity

Let $p = p(n) = \frac{\log n + c_n}{n}$.

$\mathcal{C}$: G does not have isolated vertices.

$$\lim_{n\to\infty} \mathbb{P}(\mathcal{G}(n,p) \in \mathcal{C}) = \begin{cases} 0 & \text{if } c_n \to -\infty \\ e^{-e^{-c}} & \text{if } c_n \to c \\ 1 & \text{if } c_n \to \infty. \end{cases}$$

Moreover,

$$\mathbb{P}(\mathcal{G}(n,p) \text{ is connected}) = \mathbb{P}(\mathcal{G}(n,p) \in \mathcal{C}) + o(1).$$

Trivial bottleneck (isolated vertices) is the only bottleneck.

k-connectivity

G is k-connected if the removal of at most $k-1$ vertices of G does not disconnect it.

Theorem (Erdös and Rényi, 1961)

Fix $k \in \mathbb{N}$. *Let* $p = p(n) = \frac{\log n + (k-1)\log\log n + c_n}{n}$. *Then,*

$$\lim_{n\to\infty} \mathbb{P}(\mathcal{G}(n,p) \text{ is } k\text{-connected}) = \begin{cases} 0 & \text{if } c_n \to -\infty \\ e^{-e^{-c}/(k-1)!} & \text{if } c_n \to c \\ 1 & \text{if } c_n \to \infty. \end{cases}$$

Trivial bottleneck (vertices of degree at most $k-1$) is the only bottleneck.

Hamilton Cycles

Hamilton Cycles: cycle that spans all vertices.

The precise theorem given below can be credited to Komlós and Szemerédi (1983), Bollobás (1984) and Ajtai, Komlós and Szemerédi (1985).

Theorem

Let $p = p(n) = \frac{\log n + \log\log n + c_n}{n}$. *Then,*

$$\lim_{n\to\infty} \mathbb{P}(\mathcal{G}(n,p) \text{ has a Hamilton cycle}) = \begin{cases} 0 & \text{if } c_n \to -\infty \\ e^{-e^{-c}} & \text{if } c_n \to c \\ 1 & \text{if } c_n \to \infty. \end{cases}$$

It was a difficult question but breakthrough came with the result of Pósa (1976).

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Trivial bottleneck (vertices of degree 0 or 1) is the only bottleneck.

k-regular subgraphs

$G' = (V', E')$ is a subgraph of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$.

$G' = (V', E')$ is k-regular if each vertex of G' has degree k.

Question: What is the threshold for $\mathcal{G}(n, p)$ to have k-regular subgraph (where $k \geq 3$ is a fixed integer)?

Letzter (2013) proved that this threshold is sharp. That is, there exists $r_k \in \mathbb{R}$ such that for any $\varepsilon > 0$

$$\lim_{n \to \infty} \mathbb{P}(\mathcal{G}(n, p) \text{ has } k\text{-regular subgraph}) = \begin{cases} 0 & \text{if } pn \leq r_k - \varepsilon \\ 1 & \text{if } pn \geq r_k + \varepsilon. \end{cases}$$

Question: Find (or estimate) r_k.

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k-regular subgraphs and k-cores

Fix $k \in \mathbb{N}$. The k-core of a graph $G = (V, E)$ is the largest set $S \subseteq V$ such that the minimum degree δ_S in the induced subgraph $G[S]$ is at least k.

This is unique because if $\delta_S \geq k$ and $\delta_T \geq k$, then $\delta_{S \cup T} \geq k$.

$r_k \geq c_k$, where c_k is the threshold for the appearance of a subgraph with minimum degree at least k; that is, a non-empty k-core.

The k-core of a graph can be found be repeatedly deleting vertices of degree less than k from the graph.

For $k \geq 3$, a.a.s. either there is no k-core in $\mathcal{G}(n, p)$ or one of linear size (Łuczak, 1991).

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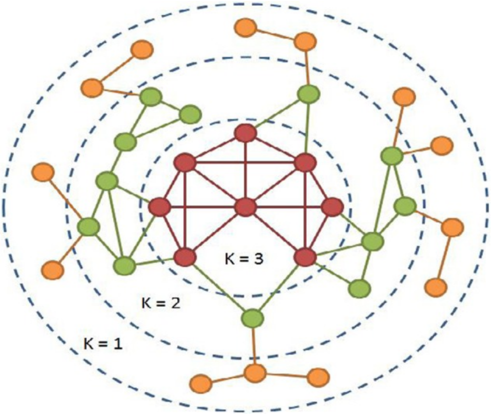
K = 3
K = 2
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The precise size and first occurrence of k-cores for $k \geq 3$ was established by Pittel, Spencer, and Wormald (1996).

$$c_k = \min_{x>0} \frac{x}{1 - e^{-x} \sum_{i=0}^{k-2} \frac{x^i}{i!}}.$$

Prałat, Verstraëte, and Wormald (2011) determined the asymptotic value of c_k up to an additive $O(1/\log k) = o_k(1)$ term. Setting $q_k = \log k - \log(2\pi)$, we have

$$\begin{aligned} r_k \geq c_k &= k + (kq_k)^{1/2} + \left(\frac{k}{q_k}\right)^{1/2} + \frac{q_k - 1}{3} + O\left(\frac{1}{\log k}\right) \\ &= k + \sqrt{k \log k} + O\left(\sqrt{\frac{k}{\log k}}\right). \end{aligned}$$

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Contradicting conjectures

Question: Is the threshold for a k-regular subgraph equal to the k-core threshold?

Bollobás, Kim, and Verstraëte (2006): “No” for $k = 3$ and conjectured that it is “No” for all $k \geq 4$.

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Known upper bounds and the result

Is there any upper bound for r_k (for large k)?

Bollobás, Kim, and Verstraëte (2006): $r_k \leq c \approx 4k \approx c_k + 3k$.

Prałat, Verstraëte, and Wormald (2011): the $(k+2)$-core of $\mathcal{G}(n,p)$ (if it is non-empty) contains a k-regular spanning subgraph (k-factor); that is, $r_k \leq c_{k+2} \approx c_k + 2$.

Chan and Molloy (2012) proved the same for the $(k+1)$-core; that is, $r_k \leq c_{k+1} \approx c_k + 1$.

Mitsche, Molloy, and Prałat (2018+) reduced this bound to within an exponentially small distance (as a function of k) from c_k: $r_k \leq c_k + \exp(-k/300)$.

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(Breakthrough: apply a classic theorem of Tutte to show that the $(k+2)$-core has a spanning k-regular subgraph.)

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(Breakthrough: stripping the k-core down to something to which Tutte's theorem can be applied to.)

New arguments

Observation: k-core cannot have a k-factor; for example, a.a.s. it has many vertices of degree $k+1$ whose neighbours all have degree k.

New arguments required in this work are:

(i) stripping the k-core down to something to which Tutte's theorem can be applied to (requires a delicate variant of the configuration model).

(ii) applying Tutte's theorem to it (the presence of degree k vertices brings new challenges).

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Contradiction with the result of Gao?

The number of problematic vertices is linear in n. Removing them from the k-core will cause a linear number of vertices to have their degrees drop below k.

If c is too close to c_k, then a.a.s. what remains will have no k-core: c has to be bounded away from c_k.

The number of problematic vertices is very small: $e^{-\Theta(k)}n$. So we only need c to be bounded away from c_k by $e^{-\Theta(k)}$.

The subgraph that we show to have a k-factor consists of all but $e^{-\Theta(k)}n$ vertices of the k-core. This is consistent with a result of Gao (2014) who proved that any k-regular subgraph must contain all but at most $\varepsilon_k n$ vertices of the k-core where $\varepsilon_k \to 0$ as k grows.

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The number of problematic vertices is linear in n. Removing them from the k-core will cause a linear number of vertices to have their degrees drop below k.

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Tutte's theorem

Γ: graph with minimum degree at least k.
$L = L(\Gamma)$: vertices v with $d_\Gamma(v) = k$ (low vertices of Γ).
$H = H(\Gamma)$: vertices v with $d_\Gamma(v) \geq k+1$ (high vertices of Γ).
We use Z_L, Z_H to denote $Z \cap L$, respectively $Z \cap H$.
$e(S)$: the number of edges of Γ with both endpoints in S.
$e(S,T)$: the number of edges of Γ from S to T.
$q(S,T)$: the number of components Q of $H \setminus (S \cup T)$ such that $k|Q|$ and $e(Q,T)$ have different parity.

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Tutte's theorem: Γ has a k-factor if and only if for every pair of disjoint sets $S, T \subseteq V(\Gamma)$,

$$k|S| \geq q(S, T) + k|T| - \sum_{v \in T} d_{\Gamma \setminus S}(v).$$

(In fact, the result was initially proved by Belck in 1950.)

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We used the following consequence of Tutte's theorem:
Γ has a k-factor if for every pair of disjoint sets $S, T \subseteq V(\Gamma)$,

$$k|S| + \sum_{v \in T_H} (d_\Gamma(v) - k) \geq q(S,T) + e(S,T).$$

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In fact, in all but one case we check the stronger condition:
Γ has a k-factor if for every pair of disjoint sets $S, T \subseteq V(\Gamma)$,

$$k|S| + |T_H| \geq q(S, T) + e(S, T).$$

The desired subgraph of the k-core

Our goal is to find (for k sufficiently large) a subgraph K of the k-core with the following properties:

(K1) for every vertex $v \in K$, $k \le d_K(v) \le 2k$;

(K2) for every vertex $v \in K$ with $d_K(v) \ge k+1$, we have $|\{w \in N_K(v) : d_K(w) = k\}| \le \frac{9}{10}k$;

(K3) $|K| \ge \frac{n}{3}$;

(K4) $k|K|$ is even.

In fact, we were able to find an induced subgraph K of G satisfying these properties.

It is easy to modify K to enforce the final property (K4), if necessary, at the end.

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Typical situation

(K2) was particularly challenging to enforce.

Typical approach:
(i) keep removing vertices violating one of (K1-3);
(ii) the remaining graph is uniformly random conditional on its degree sequence (for example, this happens when analyzing the k-core stripping process).

In some situations:
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Our situation

In our situation, enforcing (K2) requires conditioning on the number of remaining neighbours each vertex has in W, the set of vertices of degree k. Unfortunately, W changes during the process!

We partition the vertex set (in the remaining graph) into:
W_0: the vertices that had degree k in the k-core
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Note that vertices may move from R to W_1 during our procedure, but no vertex leaves W_0 unless it is deleted.

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R: the vertices of degree greater than k.

W_1 is much smaller than W_0 and so we can afford to delete vertices if they have at least two neighbours in W_1 rather than at least $\frac{9}{10}k$. This simpler deletion rule helps us deal with the fact that W_1 is changing throughout our stripping process.

STRIP algorithm

We say a vertex v is deletable if in the initial k-core:
(D1) $\deg(v) > 2k$;
(D2) $v \notin W_0$ (that is, $\deg(v) \geq k+1$) and v has at least $\frac{1}{2}k$ neighbours in W_0;

or if in the remaining graph:
(D3) $\deg(v) < k$;
(D4) $v \in R$ and v has at least two neighbours that are in W_1; or
(D5) $v \in W_1$ and v has a neighbour that is either (i) in R and deletable, or (ii) in W_1.

Furthermore,
(D6) once a vertex becomes deletable it remains deletable.

STRIP algorithm

Q: the set of deletable vertices.
$\beta = e^{-k/200}$.

1. Begin with the k-core, and initialize Q to be all vertices v with $\deg(v) > 2k$ or $v \notin W_0$ and v has at least $\frac{1}{2}k$ neighbours in W_0.
2. Until $Q = \emptyset$ or until we have run βn iterations, let v be the next vertex in Q, according to a specific fixed vertex ordering. Let N be the set of neighbours of v.
   1. Remove v from the graph (and from Q).
   2. If any $u \in N$ that is in R now has degree at most k, then move u from R to W_1.
   3. If any vertex $w \notin Q$ is now deletable, place w into Q.

Additional expansion properties

There exist constants $\gamma, \epsilon_0 > 0$, $k_0 \in \mathbb{N}$ such that for any $k \geq k_0$, a.a.s. K satisfies:

(P1) For every $Y \subseteq V(K)$ with $|Y| \leq 10\epsilon_0 n$, $e(Y) < \frac{k|Y|}{6000}$.

(P2) For every $Y \subseteq V(K)$ with $|Y| \leq \frac{1}{2} V(K)$, $e(Y, V(K) \setminus Y) \geq \gamma k |Y|$.

(P3) For every disjoint pair of sets $X, Y \subseteq V(K)$ with $|X| \geq \frac{1}{200}|Y|$ and $|Y| \leq \epsilon_0 n$, $e(X, Y) < \frac{1}{2}\gamma k |X|$.

(P4) For every disjoint pair of sets $X, Y \subseteq V(K)$ with $|X| + |Y| \leq \epsilon_0 n$, $e(X, Y) < \left(1 + \frac{1}{2000}\right) |N(X) \cap Y| + \frac{k}{100}|X|$.

(P5) For every disjoint pair of sets $S, T \subseteq V(K)$ with $|T| < \frac{1}{10}\epsilon_0 n$ and $|S| > \frac{9}{10}\epsilon_0 n$, $e(S, T) < \frac{3}{4} k |S|$.

(P6) For every disjoint pair of sets $S, T \subseteq V(K)$ with $|T| \geq \frac{1}{10}\epsilon_0 n$, we have $e(S, T) \leq k|S| + \frac{3}{4}\sqrt{k \log k}|T|$ and $\sum_{v \in T} d(v) > (k + \frac{7}{8}\sqrt{k \log k})|T|$.

Conclusion

A.a.s. STRIP halts with $Q = \emptyset$ within βn iterations.
(17.5 pages!)

Enforcing (K4).
(half a page)

Checking (P1-6).
(3 pages + PVW + CM)

Verifying (K1–4,P1–6) implies Tutte's condition.
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Thank you!