RADICALS OF SKEW INVERSE LAURENT SERIES RINGS

A. ALHEVAZ[1] **AND D. KIANI**[1,2]

[1] Department of Pure Mathematics, Faculty of Mathematics and Computer Science,
Amirkabir University of Technology (Tehran Polytechnic),
P.O. Box: 15875-4413, Tehran, Iran.
[2] School of Mathematics, Institute for Research in Fundamental Sciences (IPM),
P.O. Box: 19395-5746, Tehran, Iran.[1]

Abstract. In this note, we continue to study zero-divisor properties of skew inverse Laurent series rings $R((x^{-1};\sigma,\delta))$, where R is an associative ring equipped with an automorphism σ and a σ-derivation δ. We first introduce (σ,δ)-SILS Armendariz rings, a generalization of the standard Armendariz condition from ordinary polynomial ring $R[x]$ to skew inverse Laurent series ring $R((x^{-1};\sigma,\delta))$. We study the ring-theoretical properties of (σ,δ)-SILS Armendariz rings and using the properties of these rings, we characterize radicals of the skew inverse Laurent series ring $R((x^{-1};\sigma,\delta))$, in terms of a (σ,δ)-SILS Armendariz ring R. Also, we prove that several properties transfer between R and $R((x^{-1};\sigma,\delta))$, in case R is an σ-compatible (σ,δ)-SILS Armendariz ring.

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1. Introduction

Throughout this paper, R denotes an associative ring with identity, σ an automorphism of R and δ a σ-derivation of a ring R (i.e., δ is an additive operator on R with the property that $\delta(ab)=\delta(a)b+\sigma(a)\delta(b)$. Then we denote by $R((x^{-1};\sigma,\delta))$ the skew inverse Laurent series ring over the coefficient ring R formed by formal series $f(x)=\sum_{i=-\infty}^{m}a_ix^i$, where x is a variable, m is an integer (maybe negative), and the coefficients a_i of the series f are elements of the ring R. In the ring $R((x^{-1};\sigma,\delta))$, addition is defined as usual and multiplication is defined with respect to the relations

$$xa=\sigma(a)x+\delta(a),\;\; x^{-1}a=\sum_{i=0}^{\infty}\sigma^{-1}(-\delta\sigma^{-1})^{i-1}(a)x^{-i}.$$

Skew inverse Laurent series rings have wide applications. Not only do they provide interesting examples in non-commutative algebra, they have also been a valuable tool used first by Hilbert in the study of the independence of geometry axioms.

[1]Corresponding author. dkiani@aut.ac.ir
a.alhevaz@aut.ac.ir and a.alhevaz@yahoo.com.

The ring-theoretical properties of skew inverse Laurent series rings have been investigated by many authors (see [18], [31] and [37-39], for instance). Some of these have worked either with the case $\delta = 0$ or the case where σ is the identity. With the impetus of quantized derivations, renewed interest in the general skew inverse Laurent series ring $R((x^{-1};\sigma,\delta))$ has arisen during the last few years. For the continuation of ring-theoretical properties of skew inverse Laurent series rings, in this paper, we study the relationship between zero-divisor properties of a ring R and the general skew inverse Laurent series ring $R((x^{-1};\sigma,\delta))$.

A ring R is said to be *Armendariz* if the product of two polynomials in $R[x]$ is zero if and only if the product of their coefficients is zero. This definition was coined by Rege and Chhawchharia [36] in recognition of Armendariz's proof in [6, Lemma 1] that *reduced* rings (i.e., rings without non-zero nilpotent elements) satisfy this condition. The more comprehensive study of Armendariz rings was carried out recently. As observed by Hirano in [25], the Armendariz condition hides a remarkable connection between the set of annihilators of R and those of $R[x]$. Namely, the Armendariz rings are precisely those rings R for which there is a bijective correspondence between the right annihilators of R and the right annihilators of $R[x]$. Several papers are devoted to studying the Armendariz property of rings (see the references for some literature on the subject). Following [23], an endomorphism σ of a ring R is called *compatible* if for all $a, b \in R$, $ab = 0 \Leftrightarrow a\sigma(b) = 0$. A ring R is called σ-compatible, if there exist a compatible endomorphism σ of R. Also by [28], an endomorphism σ of a ring R is called *rigid* if for every $a \in R$, $a\sigma(a) = 0 \Leftrightarrow a = 0$.

In this paper, first we apply the concept of Armendariz ring to skew inverse Laurent series ring over general non-commutative ring. We say R is an (σ,δ)-*SILS Armendariz ring*, if for each elements $f(x) = \sum_{i=-\infty}^{m} a_i x^i$ and $g(x) = \sum_{j=-\infty}^{n} b_j x^j \in R((x^{-1};\sigma,\delta))$, $f(x)g(x) = 0$ implies that $a_i x^i b_j x^j = 0$, for each $i \leq m$ and $j \leq n$. Although reduced rings are (σ,δ)-SILS Armendariz for any compatible automorphism σ of R, but we will provide a fairly rich classes of non-reduced (σ,δ)-SILS Armendariz rings. An equivalent characterization of an (σ,δ)-SILS Armendariz ring is given, which is useful to simplify the proofs. Then, we are concerned with the characterization of the radicals of skew inverse Laurent series ring $R((x^{-1};\sigma,\delta))$, in terms of a (σ,δ)-SILS Armendariz ring R. When R is an σ-compatible (σ,δ)-SILS Armendariz ring, then the sum of all nilpotent ideals of R coincides with the sum of all nil left ideals of R. In spite of the many great advances made in ring theory in recent times, Köthe's Conjecture, which posits that a ring with no non-zero nil ideals has no non-zero nil one-sided ideals, has remained unsolved in general. For several special classes of rings, the conjecture has been shown to be true. We will presently add σ-compatible (σ,δ)-SILS Armendariz rings to this list. There is considerable interest in studying if and how certain properties of rings are preserved under skew inverse Laurent series extensions: if a ring R has some property, one would like to know whether the ring $R((x^{-1};\sigma,\delta))$ also enjoys that property. We prove that several properties transfer between R and $R((x^{-1};\sigma,\delta))$, in case R is an σ-compatible (σ,δ)-SILS Armendariz ring.

2. Armendariz Property of Skew Inverse Laurent Series Rings

There are many ways to generalize Armendariz's result. Our goal in this section is to see to what extent we can generalize the results obtained about the standard Armendariz condition from polynomial ring $R[x]$ to the full generality of the skew inverse Laurent series ring $R((x^{-1};\sigma,\delta))$. Because of the complexity of the coefficients that arise upon multiplication in $R((x^{-1};\sigma,\delta))$, this generalization is fraught with difficulties.

We apply the concept of Armendariz ring to skew inverse Laurent series ring over general non-commutative ring. We study ring-theoretical properties of (skew-formal) inverse Laurent series rings, which turn out to be particularly well behaved analogues of commutative Laurent series rings. These rings provide non-commutative generalizations of commutative Laurent series rings.

We begin by giving a series of definitions with the aim of producting generalizations of (linearly) Armendariz condition in the context of skew inverse Laurent series rings.

Definition 2.1. (i) Let σ be an automorphism and δ a σ-derivation of a ring R. The ring R is called a (σ,δ)-*skew Armendariz ring of skew inverse Laurent series type* (briefly, (σ,δ)-*SILS Armendariz ring*), if for each elements $f(x)=\sum_{i=-\infty}^{m} a_i x^i$ and $g(x)=\sum_{j=-\infty}^{n} b_j x^j \in R((x^{-1};\sigma,\delta))$, $f(x)g(x)=0$ implies $a_i x^i b_j x^j=0$, for each $i\leq m$ and $j\leq n$.

(ii) Let σ be an automorphism and δ a σ-derivation of a ring R. The ring R is called a *linearly* (σ,δ)-*SILS Armendariz ring*, if for each elements $f(x)=a_{-1}x^{-1}+a_0$ and $g(x)=b_{-1}x^{-1}+b_0 \in R((x^{-1};\sigma,\delta))$, $f(x)g(x)=0$ implies $a_i x^i b_j x^j=0$, for $i,j\in\{-1,0\}$.

The following example shows that there exists an Armendariz ring with an automorphism σ and an σ-derivation δ which is not (linearly) (σ,δ)-SILS Armendariz.

Example 2.2. Let S be any non-zero reduced ring. Suppose $R=S\oplus S$ with the usual addition and multiplication. Then R is redued and so is Armendariz. Let $\sigma: R\to R$ be an automorphism defined by $\sigma((a,b))=(b,a)$ and $\delta: R\to R$ be an σ-derivation defined by $\delta((a,b))=(a-b,0)$. It is easy to see that $\sigma^2=id_R$ and $\delta^2=\delta$. Let $f(x)=(0,1)x^{-1}-(0,1)$, $g(x)=(0,1)x^{-1}+(1,0)\in R((x^{-1};\sigma,\delta))$. It is not hard to see that $(0,1)x^{-1}(0,1)x^{-1}=(0,0)$ and $(0,1)x^{-1}(1,0)=(0,1)x^{-1}$. Then we can see that $f(x)g(x)=0$, but since $(0,1)x^{-1}(1,0)=(0,1)x^{-1}$, the ring R is not (linearly) (σ,δ)-SILS Armendariz.

It will be useful to establish a criteria for transfer of the (σ,δ)-SILS Armendariz condition from one ring to another.

Proposition 2.3. *Let σ be an automorphism and δ a σ-derivation of a ring R. Let S be a ring and $\gamma: R\to S$ a ring isomorphism. Then R is (σ,δ)-SILS Armendariz if and only if S is $(\gamma\sigma\gamma^{-1},\gamma\delta\gamma^{-1})$-SILS Armendariz.*

Proof. Let $\sigma' = \gamma\sigma\gamma^{-1}$ and $\delta' = \gamma\delta\gamma^{-1}$. Clearly, σ' is an automorphism of S. Also $\delta'(ab) = \gamma\delta(\gamma^{-1}(a)\gamma^{-1}(b)) = \gamma[(\delta\gamma^{-1})(a)\gamma^{-1}(b) + (\sigma\gamma^{-1})(a)(\delta\gamma^{-1}(b))] = \delta'(a)b + \sigma'(a)\delta'(b)$. Thus δ' is a σ'-derivation on S. Suppose that $a' = \gamma(a)$ and $b' = \gamma(b)$, for each $a, b \in R$. Note that $\gamma(a\sigma^k\delta^t(b)) = a'\gamma(\sigma^k\delta^t(b)) = a'\gamma(\sigma^k\gamma^{-1}\gamma\delta^t\gamma^{-1}\gamma(b)) = a'(\gamma\sigma\gamma^{-1})^k(\gamma\delta\gamma^{-1})^t(b') = a'\sigma'^k\delta'^t(b')$. Also $f(x)g(x) = 0$ in $R((x^{-1};\sigma,\delta))$ if and only if $f'(x)g'(x) = 0$ in $S((x^{-1};\sigma',\delta'))$. On the other hand, $a_ix^ib_jx^j = 0$, for each i, j if and only if $a'_ix^ib'_jx^j = 0$, for each i, j. Thus R is (σ,δ)-SILS Armendariz if and only if S is $(\gamma\sigma\gamma^{-1}, \gamma\delta\gamma^{-1})$-SILS Armendariz.

We recall the definition of a compatible endomorphism from [23]. An endomorphism σ of a ring R is called *compatible* if for all $a, b \in R$, $ab = 0 \Leftrightarrow a\sigma(b) = 0$. A ring R is called σ-compatible, if there exist a compatible endomorphism σ of R. Moreover, for a σ-derivation δ of R, the ring is said to be δ-compatible if for each $a, b \in R$, $ab = 0 \Rightarrow a\delta(b) = 0$. A ring R is (σ,δ)-compatible if it is both σ-compatible and δ-compatible. We will also want to consider a condition on endomorphism stronger than compatibility, namely the *rigidity* condition studied in [28]. An endomorphism σ of a ring R is called *rigid* if for every $a \in R$, $a\sigma(a) = 0 \Leftrightarrow a = 0$. Basic properties of rigid and compatible endomorphisms, proved by Hashemi and Moussavi in [23], are summarized here:

Lemma 2.4. *Let σ be an endomorphism of a ring R. Then*:
(i) *if σ is compatible, then σ is injective*;
(ii) *σ is compatible if and only if for all $a, b \in R$, $\sigma(a)b = 0 \Leftrightarrow ab = 0$*;
(iii) *the following conditions are equivalent*;
  (1) *σ is rigid*;
  (2) *σ is compatible and R is reduced*;
  (3) *for every $a \in R$, $\sigma(a)a = 0$ implies that $a = 0$.*

Before we record our first main result about the (σ,δ)-SILS Armendariz rings, let us note a couple of observations concerning the notion of a δ-compatible ring. These facts will be used freely without mention in what follows.

Lemma 2.5. *Let σ be an automorphism and δ a σ-derivation of a ring R. Then*:
  (1) *Each σ-compatible (σ,δ)-SILS Armendariz ring is δ-compatible.*
  (2) *Each reduced ring is δ-compatible.*

Proof. (1) Let R be a σ-compatible (σ,δ)-SILS Armendariz ring and $a, b \in R$ such that $ab = 0$. Then we have $\sigma^{k_1}(a_1)\sigma^{k_2}(a_2) = 0$ for each integers k_1 and k_2, since σ is a compatible automorphism. On the other hand, from $ab = 0$ we have $\delta(ab) = \delta(a)b + \sigma(a)\delta(b) = 0$. Take $f(x) = \delta(a) + \sigma(a)x$ and $g(x) = b + bx \in R((x^{-1};\sigma,\delta))$. Then $f(x)g(x) = 0$ and hence $\delta(a)b = 0$, since R is (σ,δ)-SILS Armendariz. So from $\delta(ab) = \delta(a)b + \sigma(a)\delta(b) = 0$, we have $\sigma(a)\delta(b) = 0$. Since σ is compatible we have $a\delta(b) = 0$, as desired.

(2) Let R be a reduced ring and $a, b \in R$ such that $ab = 0$. Then $\delta(ab) = \delta(a)b + a\delta(b) = 0$. Multiplying a from right-hand side of the above, we have $\delta(a)ba + a\delta(b)a = 0$ and hence $a\delta(b)a = 0$, since $ba = 0$. So $a\delta(b)a\delta(b) = 0$ and hence $a\delta(b) = 0$, since R is reduced.

We shall now derive the condition for a ring R to be (σ,δ)-SILS Armendariz. We prove that, (σ,δ)-SILS Armendariz rings is a fairly big class which includes for instance σ-rigid rings.

Theorem 2.6. *For each automorphism σ of R and any σ-derivation δ, every σ-rigid ring is (σ,δ)-SILS Armendariz.*

Proof. Let R be a σ-rigid ring. Then σ is a compatible automorphism of R, by Lemma 2.4. First notice that by a similar way as used in [23, Lemma 3.2], we can prove that if $ab=0$, then $a\sigma^k(b)=\sigma^k(a)b=0$ and also $\sigma^k(a)\delta^l(b)=\delta^l(a)\sigma^k(b)=0$ for each integer k and every positive integer l. Also, if $\sigma^k(a)b=0$ for some integer k, then $ab=0$. Now, let $f(x)g(x)=0$, where $f(x)=\sum_{i=-\infty}^{m} a_ix^i$ and $g(x)=\sum_{j=-\infty}^{n} b_jx^j \in R((x^{-1};\sigma,\delta))$. We show that $a_ib_j=0$, for each i and j. We work by induction on $k=i+j$. If $k=m+n$, then $i=m$ and $j=n$. We know $a_m\sigma^m(b_n)=0$ by looking at the degree $m+n$ term in $f(x)g(x)=0$. Thus $a_mb_n=0$, since σ is compatible. Assume that for $i+j>k$, $a_ib_j=0$. From the degree k coefficient in $f(x)g(x)=0$ we obtain $\sum_{t=0}^{k} a_t\sigma^t(b_{k-t})+\omega_k=0$, where ω_k is a sum of some terms of the forms $a_i\delta^t(b_j)$, $a_i\delta\alpha(b_j)$ and $a_i\alpha\delta(b_j)$ such that $i+j>k$. Since for each $i+j>k$ we assume that $a_ib_j=0$, so by Lemma 2.5(2), we have $\omega_k=0$. Hence $\sum_{t=0}^{k} a_t\sigma^t(b_{k-t})=0$. By multiplying b_k from the left-hand side, we obtain $b_ka_0b_k=0$ and hence $a_0b_k=0$, since R is σ-rigid and also for each $i+j>k$ we assume that $a_ib_j=0$. Now we have $\sum_{t=1}^{k} a_t\sigma^t(b_{k-t})=0$. Similar above, by multiplying b_{k-1} from the left-hand side, we obtain $b_{k-1}a_1\sigma(b_{k-1})=0$ and hence $a_1b_{k-1}=0$, since R is σ-rigid and also for each $i+j>k$ we assume that $a_ib_j=0$. Continuing in this process, we get $a_ib_j=0$ for each i,j with $i+j=k$, as desired.

Corollary 2.7. *Every reduced ring with any derivation δ is δ-SILS Armendariz.*

The next example shows that without compatibility condition, Theorem 2.6 is not true in general.

Example 2.8. Keeping all of the notations from the Example 2.2, the ring $R=S\oplus S$ is reduced. We have $(1,0)(0,1)=0$, but $(1,0)\sigma((0,1))=(1,0)$ and $(1,0)\delta((0,1))=(-1,0)$. Thus R is neither σ-compatible nor δ-compatible. Also by Example 2.2, R is not (linearly) (σ,δ)-SILS Armendariz.

A ring R is *semi-commutative* if the right annihilator of each element of R is an ideal (equivalently, if for all $a,b\in R$ we have $ab=0\Rightarrow aRb=0$). A ring R is *symmetric* if for all $a,b,c\in R$ we have $abc=0\Rightarrow bac=0$. A ring R is called *reversible* if for all $a,b\in R$ we have $ab=0\Rightarrow ba=0$. Recall that a ring R is *Abelian* if every idempotent of R is central. Note that every reduced ring is symmetric, every symmetric ring is reversible, every reversible ring is semi-commutative and every semi-commutative ring is Abelian.

In the following proposition, we determine the idempotents of $R((x^{-1};\sigma,\delta))$ in terms of the idempotents of R. Also, we show that each (σ,δ)-SILS Armendariz ring is Abelian.

Proposition 2.9. *Let R be an (σ,δ)-SILS Armendariz ring. Then we have the following statements*:

(i) $\sigma(e)=e$ *and* $\delta(e)=0$*, for each* $e^2=e\in R$.

(i) *If* $e^2=e\in R((x^{-1};\sigma,\delta))$*, then* $e\in R$.

(iii) R *is an Abelian ring.*

(iv) $R((x^{-1};\sigma,\delta))$ *is an Abelian ring.*

Proof. (i) Let $e^2=e\in R$. Then we have $\delta(e)=\delta(e^2)=\delta(e)e+\sigma(e)\delta(e)$. Now suppose that $f(x)=\delta(e)+\sigma(e)x$ and $g(x)=(e-1)+(e-1)x\in R((x^{-1};\sigma,\delta))$. Then we have $f(x)g(x)=0$. Since R is an (σ,δ)-SILS Armendariz ring, $\delta(e)e=\delta(e)$ and hence $\sigma(e)\delta(e)=0$. On the other hand, if we take $p(x)=\delta(e)-(1-\sigma(e))x$ and $q(x)=e+ex\in R((x^{-1};\sigma,\delta))$, then we have $p(x)q(x)=0$. Thus $\delta(e)=\delta(e)e=0$, since R is an (σ,δ)-SILS Armendariz ring. Now take $t(x)=(1-e)+(1-e)\sigma(e)x$ and $u(x)=e+(e-1)\sigma(e)x\in R((x^{-1};\sigma,\delta))$. Then $t(x)u(x)=0$ and hence $\sigma(e)=e\sigma(e)$, since R is an (σ,δ)-SILS Armendariz ring. On the other hand, $v(x)w(x)=0$, where $v(x)=e+e(1-\sigma(e))x$ and $w(x)=(1-e)-e(1-\sigma(e))x\in R((x^{-1};\sigma,\delta))$. Now since R is (σ,δ)-SILS Armendariz, we have $e=e\sigma(e)$. Therefore $e=\sigma(e)$, as desired.

(ii) Let $e=\sum_{i=-\infty}^{m}e_ix^i$ be an idempotent of $R((x^{-1};\sigma,\delta))$. Since $(1-e)e=0$, we have $(1-e_0)e_i=0$, for each i. Thus $e_i=e_0e_i$, for each i. On the other hand, since $e(1-e)=0$, we have $e_0(1-e_0)=0$ and $e_0e_i=0$, for each $i\neq 0$. Thus $e_i=0$, for each $i\neq 0$. Hence $e=e_0\in R$, as desired.

(iii) Let R be (σ,δ)-SILS Armendariz ring, $e^2=e\in R$ and $r\in R$. Suppose that $f(x)=e-er(1-e)x$ and $g(x)=(1-e)+er(1-e)x\in R((x^{-1};\sigma,\delta))$. Then we have $f(x)g(x)=0$ and hence $er(1-e)=0$, since R is (σ,δ)-SILS Armendariz ring. So $er=ere$. On the other hand, $h(x)k(x)=0$, where $h(x)=(1-e)-(1-e)rex$ and $k(x)=e+(1-e)rex\in R((x^{-1};\sigma,\delta))$. Since R is an (σ,δ)-SILS Armendariz ring, it implies that $(1-e)(1-e)re=0$. Therefore $re=ere$ and so $re=er$ which implies that R is Abelian.

(iv) It follows by (i), (ii) and (iii).

Theorem 2.10. *Let R be a ring, σ a compatible automorphism and δ an σ-derivation of R. Then the following statements are equivalent*:

(1) *R is (σ,δ)-SILS Armendariz ring.*

(2) *For each* $f(x)=\sum_{i=-\infty}^{m}a_ix^i$ *and* $g(x)=\sum_{j=-\infty}^{n}b_jx^j\in R((x^{-1};\sigma,\delta))$, $f(x)g(x)=0$ *implies* $a_0b_j=0$*, for each* $j\leq n$.

(3) *For each* $f(x)=\sum_{i=-\infty}^{m}a_ix^i$ *and* $g(x)=\sum_{j=-\infty}^{n}b_jx^j\in R((x^{-1};\sigma,\delta))$, $f(x)g(x)=0$ *implies* $a_ib_j=0$*, for each* $i\leq m$ *and* $j\leq n$.

Proof. (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) are clear, and we only prove the (2) $\Rightarrow$ (3). Let $f(x)=\sum_{i=-\infty}^{m}a_ix^i$ and $g(x)=\sum_{j=-\infty}^{n}b_jx^j\in R((x^{-1};\sigma,\delta))$ and $f(x)g(x)=0$. First, we show that $a_1b_j=0$, for all $j\leq n$. Note that since $f(x)g(x)=0$, we have

$$\left(\sum_{i=-\infty}^{m}a_ix^{i-1}\right)x\left(\sum_{j=-\infty}^{n}b_jx^j\right)=\left(\sum_{i=-\infty}^{m}a_ix^{i-1}\right)\left(\sum_{j=-\infty}^{n+1}(\sigma(b_{j-1})+\delta(b_j))x^j\right)=0.$$

So, from (2) we have $a_1\sigma(b_n)=0$ and since σ is compatible, we have $a_1b_n=0$. So $0=\delta(a_1b_n)=\delta(a_1)b_n+\sigma(a_1)\delta(b_n)$. Now let $p(x)=\delta(a_1)+\sigma(a_1)x$ and $q(x)=b_n+b_nx$. Then $p(x)q(x)=\delta(a_1)b_n+\sigma(a_1)\delta(b_n)+[\delta(a_1)b_n+\sigma(a_1)\sigma(b_n)+\sigma(a_1)\delta(b_n)]x+$

$[\sigma(a_1)\sigma(b_n)]x^2 = 0$. Hence $\delta(a_1)b_n = 0$ by (2) and hence $\sigma(a_1)\delta(b_n) = 0$. Since R is σ-compatible, we have $a_1\delta(b_n) = 0$. On the other hand, since $a_1(\sigma(b_{n-1})+\delta(b_n)) = 0$, we have $a_1\sigma(b_{n-1}) = 0$ and then $a_1b_{n-1} = 0$, since σ is compatible. Continuing in this process, $a_1b_j = 0$ for all $j \leq n$. Similarly, we can prove that $a_ib_j = 0$, for each $i \geq 0$ and $j \leq n$.

Next, we show that $a_ib_j = 0$, for each $i \leq 0$ and $j \leq n$. Note that $0 = f(x)g(x) = (\sum_{i=-\infty}^{m} a_ix^{i+1})x^{-1}(\sum_{j=-\infty}^{n} b_jx^j) = (\sum_{i=-\infty}^{m} a_ix^{i+1})(\sum_{j=-\infty}^{n} c_jx^{j-1})$, where $c_j = \sum_{k=0}^{n-j} \sigma^{j+k-n-1}(b_{n-k})(-\delta)^{n-j-k}(b_{n-k})$. So, by (2) we have $a_{-1}\sigma^{-1}(b_n) = 0$ and hence $a_{-1}b_n = 0$, since σ is a compatible automorphism. By a similar way as used in the above, we can show that $a_{-1}b_j = 0$, for each $j \leq n$. Similarly, we can prove that $a_ib_j = 0$, for each $i \leq 0$ and $j \leq n$, as desired.

Theorem 2.11. *Let R be an σ-compatible (σ,δ)-SILS Armendariz ring. If $a, b \in R$ and $c^n = 0$ for some positive integer n, then $ab = 0$ implies $acb = 0$.*

Proof. First we prove that if $ab = c^n = 0$ for some positive integer n, then $ac^{n-1}b = 0$, for some $a, b, c \in R$. The case $n = 1$ is clear. Now assume that $n \geq 2$ and take $f(x) = a - ac^{n-1}x$ and $g(x) = b + c^{n-1}\delta(b) + c^{n-1}\sigma(b)x \in R((x^{-1};\sigma,\delta))$. Then we have $f(x)g(x) = ab + ac^{n-1}\delta(b) + ac^{n-1}\sigma(b)x - ac^{n-1}\delta(b) - ac^{n-1}\sigma(b)x - ac^{n-1}\sigma(c^{n-1}\delta(b))x - ac^{n-1}\delta(c^{n-1}\delta(b)) - ac^{n-1}\delta(c^{n-1}\sigma(b))x - ac^{n-1}\sigma(c^{n-1}\sigma(b))x^2 = -ac^{n-1}\delta(c^{n-1}\delta(b)) - ac^{n-1}\sigma(c^{n-1}\delta(b))x - ac^{n-1}\delta(c^{n-1}\sigma(b))x - ac^{n-1}\sigma(c^{n-1}\sigma(b))x^2$. Since R is σ-compatible (σ,δ)-SILS Armendariz, $c^n = 0$, $n \geq 2$, and $ac^{n-1}c^{n-1}\sigma(b) = 0 = ac^{n-1}c^{n-1}\delta(b)$, then $ac^{n-1}\sigma(c^{n-1}\sigma(b)) = ac^{n-1}\sigma(c^{n-1}\delta(b)) = 0$, by compatibility of σ, and also $ac^{n-1}\delta(c^{n-1}\sigma(b)) = ac^{n-1}\delta(c^{n-1}\delta(b)) = 0$, by Lemma 2.5(1). Hence we have $f(x)g(x) = 0$. So $ac^{n-1}\sigma(b) = 0$, by Theorem 2.10, and then $ac^{n-1}b = 0$, since R is σ-compatible. Now, without loss of generality, we can assume that $n = 2^k$ and $ab = c^n = 0$, because if $n \neq 2^k$, then there exists a positive integer k such that $2^k > n$ and hence $0 = c^n = c^nc^{2^k-n} = c^{2^k}$. By above argument, since $ab = (c^{2^{k-1}})^2 = 0$, we have $ac^{2^{k-1}}b = 0$. Take $f_{2^{k-2}}(x) = a - ac^{2^{k-2}}x$ and $g_{2^{k-2}}(x) = b + c^{2^{k-2}}\delta(b) + c^{2^{k-2}}\sigma(b)x$. Then we have $f_{2^{k-2}}(x)g_{2^{k-2}}(x) = ab + ac^{2^{k-2}}\delta(b) + ac^{2^{k-2}}\sigma(b)x - ac^{2^{k-2}}\delta(b) - ac^{2^{k-2}}\sigma(b)x - ac^{2^{k-2}}\delta(c^{2^{k-2}}\delta(b)) - ac^{2^{k-2}}\sigma(c^{2^{k-2}}\delta(b))x - ac^{2^{k-2}}\delta(c^{2^{k-2}}\sigma(b))x - ac^{2^{k-2}}\sigma(c^{2^{k-2}}\sigma(b))x^2$. Since R is σ-compatible (σ,δ)-SILS Armendariz ring and $ac^{2^{k-1}}b = 0$, then $f_{2^{k-2}}(x)g_{2^{k-2}}(x) = 0$, by Lemma 2.5(1). Then by Theorem 2.10, $ac^{2^{k-2}}\sigma(b) = 0$ and hence $ac^{2^{k-2}}b = 0$, since R is σ-compatible. Now Take $f_{2^{k-3}}(x) = a - ac^{2^{k-3}}x$ and $g_{2^{k-3}}(x) = b + c^{2^{k-3}}\delta(b) + c^{2^{k-3}}\sigma(b)x$. Then we have $f_{2^{k-3}}(x)g_{2^{k-3}}(x) = ab + ac^{2^{k-3}}\delta(b) + ac^{2^{k-3}}\sigma(b)x - ac^{2^{k-3}}\delta(b) - ac^{2^{k-3}}\sigma(b)x - ac^{2^{k-3}}\delta(c^{2^{k-3}}\delta(b)) - ac^{2^{k-3}}\sigma(c^{2^{k-3}}\delta(b))x - ac^{2^{k-3}}\delta(c^{2^{k-3}}\sigma(b))x - ac^{2^{k-3}}\sigma(c^{2^{k-3}}\sigma(b))x^2$. Since we showed $ac^{2^{k-2}}b = 0$, then $f_{2^{k-3}}(x)g_{2^{k-3}}(x) = 0$, by Lemma 2.5(1). Then by Theorem 2.10, $ac^{2^{k-3}}\sigma(b) = 0$ and hence $ac^{2^{k-3}}b = 0$, since R is σ-compatible. Continuing in this way, (k-1)-times we have $ac^{2^{k-(k-1)}}b = ac^2b = 0$. Thus we have $f_1(x)g_1(x) = 0$, where $f_1(x) = a - acx$ and $g_1(x) = b + c\delta(b) + c\sigma(b)x$. So by σ-compatibility of R and Theorem 2.10, we obtain $acb = 0$, as desired.

There is another important ring-theoretic condition common in the literature related to the zero-divisor and annihilator conditions we have been studying. Faith in [12] called a ring R *right zip* if the right annihilator of a subset X of R is zero, $r_R(X) = 0$, then there exists a finite subset $Y \subseteq X$ such that $r_R(Y) = 0$; equivalently, for a left ideal L of R with $r_R(L) = 0$, there exists a finitely generated left ideal $L_1 \subseteq L$ such that $r_R(L_1) = 0$. Zelmanowitz [40] noted that any ring satisfying the descending chain condition on right annihilators is right zip, and he also showed that there exist commutative zip rings which do not satisfy the descending chain condition on (right) annihilators.

Hong et al. [26, Theorem 11], proved that an Armendariz ring R is right zip if and only if so is $R[x]$. Now we turn our attention to the relationship between the zip property of a ring R and these of the skew inverse Laurent series ring $R((x^{-1};\sigma,\delta))$. As an application of (σ,δ)-SILS Armendariz rings, we have the following theorem, that generalizes the Hong et al's result.

Theorem 2.12. *Let R be an σ-compatible (σ,δ)-SILS Armendariz ring. Then R is right zip if and only if $R((x^{-1};\sigma,\delta))$ is right zip.*

Proof. Suppose that $S = R((x^{-1};\sigma,\delta))$ is right zip. Let X be a subset of R such that $r_R(X) = 0$. If $f(x) = \sum_{i=-\infty}^{m} a_i x^i \in r_S(X)$, then $a_i \in r_R(X) = 0$ and so $f(x) = 0$. Hence $r_S(X) = 0$ and since S is right zip, there exists a finite set $Y \subseteq X$ such that $r_S(Y) = 0$. Hence $r_R(Y) = r_S(Y) \cap R = 0$. Therefore R is a right zip ring. Conversely, suppose that R is right zip and let $X \subseteq S$ such that $r_S(X) = 0$. For $f(x) = \sum_{i=-\infty}^{m} a_i x^i \in S$, C_f denotes the set of coefficients of $f(x)$, and for a subset V of S, C_V denotes the set $\bigcup_{f\in V} C_f$. Thus $r_R(C_X) = 0$, since R is σ-compatible and (σ,δ)-SILS Armendariz. Since R is right zip, there exists a finite subset $Y_0 \subseteq C_X$ such that $r_R(Y_0) = 0$. Now for each $a \in Y_0$ there exists $g_a(x) \in X$ such that at least one of the coefficients of $g_a(x)$ is a. Assume that Y be minimal between those subsets of X with the property that for each $a \in Y_0$, $g_a(x) \in Y$. Let $Y_1 = C_Y$. Then $Y_0 \subseteq Y_1$ and so $r_R(Y_1) = 0$. Now, we show that $r_S(Y) = 0$. Let $g(x) \in r_S(Y)$. Then $f(x)g(x) = 0$ for each $f(x) \in Y$. Since R is (σ,δ)-SILS Armendariz and σ-compatible, then by Theorem 2.10, $ab = 0$ for each $a \in C_f$ and $b \in C_g$. Hence $b \in r_R(Y_1) = 0$, a contradiction. Thus $g(x) = 0$ and so $r_S(Y) = 0$, as desired.

The Morita invariance of a property of R can be checked by testing if it passes to matrix rings $M_n(R)$ and corner rings eRe, with $e^2 = e$ a full idempotent ($ReR = R$). It turns out that the (linearly) (σ,δ)-SILS Armendariz property is badly behaved with regards to Morita invariance.

Example 2.13. Let R_1 be any ring and $R = M_2(R_1)$ be the 2-by-2 full matrix ring over R_1. Suppose σ be an automorphism of R, defined by $\sigma\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$. Let $f(x) = (E_{11} - E_{12})x^{-1} - E_{11}$ and $g(x) = (E_{22} - E_{12})x^{-1} + E_{22} \in R((x^{-1};\sigma))$. Then it is easy to see that $f(x)g(x) = 0$, but $(E_{11} - E_{12})x^{-1}E_{22} = -E_{12}$. Thus R is not (linearly) σ-SILS Armendariz.

Recall that for an ideal I of R, if $\sigma(I) \subseteq I$, then $\bar{\sigma} : R/I \to R/I$ defined by $\bar{\sigma}(a+I) = \sigma(a) + I$ is an endomorphism of a factor ring R/I. If σ is an automorphism and $\sigma(a) \notin I$, for each $a \in R \setminus I$, then $\bar{\sigma}$ is automorphism. The following example, shows that the class of (linearly) σ-SILS Armendariz rings are not closed under homomorphic images.

Example 2.14. Let R be the ring of quaternions with integer coefficients. Then R is a domain, and so R is σ-SILS Armendariz for any automorphism σ of R. But, R/qR is isomorphic to the 2-by-2 matrix ring over the Galois field of order q by the argument in [18, Example 2A], where q is an odd prime integer. Thus R/qR is not linearly $\bar{\sigma}$-SILS Armendariz, as shown in Example 2.13.

Clearly, if R is a domain with an automorphism σ, then R is a σ-SILS Armendariz ring. In particular, for a completely prime ideal P (i.e., if $ab \in P$, then $a \in P$ or $b \in P$), a factor ring R/P is $\bar{\sigma}$-SILS Armendariz ring. The following example shows that there exists an automorphism of an Abelian ring R whose prime radical $P(R)$ is a completely semiprime ideal and σ-SILS Armendariz ring. Also $R/P(R)$ is a $\bar{\sigma}$-SILS Armendariz ring, but R is not (linearly) σ-SILS Armendariz.

Example 2.15. Let $R = \left\{ \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \mid a - b \equiv c \equiv 0 \text{ (mod 2) and } a, b, c \in \mathbb{Z} \right\}$ and let $\sigma : R \to R$ be an automorphism defined by $\sigma\left(\begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \right) = \begin{pmatrix} a & -c \\ 0 & b \end{pmatrix}$. Clearly $P(R) = \left\{ \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} \mid c \equiv 0 \ (mod\ 2) \right\}$ is σ-SILS Armendariz ring. Also, since $P(R)$ is completely semiprime and $\sigma(P(R)) = P(R)$, then $A\sigma(A) \in P(R)$ implies that $A \in P(R)$. So $R/P(R)$ is a $\bar{\sigma}$-SILS Armendariz ring. Now we prove that R is not a (linearly) σ-SILS Armendariz ring. Let $f(x) = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} x^{-1} + \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}$ and $g(x) = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} x^{-1} + \begin{pmatrix} 0 & 2 \\ 0 & -2 \end{pmatrix} \in R((x^{-1}; \sigma))$. Then we have $f(x)g(x) = 0$, but $\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} x^{-1} \begin{pmatrix} 0 & 2 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 0 & -4 \\ 0 & 0 \end{pmatrix}$. Thus R is not a (linearly) σ-SILS Armendariz ring, as desired.

The following example shows that there exists a non-identity automorphism σ of a ring R such that I is σ-SILS Armendariz ring (as a ring without identity) and R/I is $\bar{\sigma}$-SILS Armendariz ring, for any non-zero proper ideal I of R, but R is not (linearly) σ-SILS Armendariz ring.

Example 2.16. Let F be any field and consider a ring $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. Suppose that σ be an automorphism of R defined by $\sigma\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right) = \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix}$. Let $f(x) = (E_{11} + E_{12})x^{-1} + E_{11}$ and $g(x) = (E_{12} + E_{22})x^{-1} - E_{22} \in R((x^{-1}; \sigma))$. Then it is easy to see that $f(x)g(x) = 0$, but $(E_{11} + E_{12})x^{-1}E_{22} \neq 0$ and hence R is not (linearly) σ-SILS Armendariz ring. Note that the only non-zero proper ideals of R are

$$I = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}, \; J = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix} \text{ and } \; K = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}.$$

Clearly, R/I is $\bar{\sigma}$-SILS Armendariz ring, since $R/I \cong F$. Now, we show that I is σ-SILS Armendariz ring. Assume that $F(x)G(x) = 0$, where $F(x) = \sum_{i=-\infty}^{m} A_i x^i$ and $G(x) = \sum_{j=-\infty}^{n} B_j x^j$ in $I((x^{-1};\sigma))$. Also we let $A_i = \begin{pmatrix} a_i & b_i \\ 0 & 0 \end{pmatrix}$ for each $i \leq m$ and $B_j = \begin{pmatrix} c_j & d_j \\ 0 & 0 \end{pmatrix}$ for each $j \leq n$. We know $A_m \sigma^m(B_n) = 0$ by looking at the degree $m+n$ term in $F(x)G(x) = 0$. Therefore $a_m c_n = a_m d_n = 0$. If $a_m \neq 0$, then $B_n = 0$, a contradiction; hence $a_m = 0$. This implies that $A_m \sigma^m(B_j) = 0$, for all $j \leq n$. On the other hand, by looking at the degree $m+n-1$ term in $F(x)G(x) = 0$, we have $A_{m-1}\sigma^{m-1}(B_n) + A_m \sigma^m(B_{n-1}) = 0$ and so $A_{m-1}\sigma^{m-1}(B_n) = 0$. Similar above, one can see that $a_{m-1} = 0$, since $B_n \neq 0$ and consequently $A_{m-1}\sigma^{m-1}(B_j) = 0$, for each $j \leq n$. By continuing in this way, we have $A_i \sigma^i(B_j) = 0$ and so I is σ-SILS Armendariz ring. Similarly, one can see that R/J is $\bar{\sigma}$-SILS Armendariz and J is σ-SILS Armendariz ring. Finally, it can be easily checked that K is σ-SILS Armendariz ring. Moreover, R/K is $\bar{\sigma}$-SILS Armendariz ring, since R/K is reduced and $\bar{\sigma}$ is an identity map on R/K.

We have shown that the (σ,δ)-SILS Armendariz property is badly behaved with regards to Morita invariance. One might ask whether the (σ,δ)-SILS Armendariz property passes to corner rings. In the following result, we show that this is true for central idempotents. The following is a characterization of an Abelian ring R to be (σ,δ)-SILS Armendariz in terms of its idempotents.

Theorem 2.17. *Let R be an Abelian ring, σ an automorphism and δ an σ-derivation of R. Then the following statements are equivalent*:

(i) *R is (σ,δ)-SILS Armendariz*;

(ii) *For each idempotent $e \in R$ such that $\sigma(e) = e$ and $\delta(e) = 0$, eR and $(1-e)R$ are (σ,δ)-SILS Armendariz*;

(iii) *For some idempotent $e \in R$ such that $\sigma(e) = e$ and $\delta(e) = 0$, eR and $(1-e)R$ are (σ,δ)-SILS Armendariz.*

Proof. We only need to prove the (iii) $\Rightarrow$ (i). Suppose that for some idempotent $e \in R$ such that $\sigma(e) = e$ and $\delta(e) = 0$, eR and $(1-e)R$ are (σ,δ)-SILS Armendariz and let $f(x) = \sum_{i=-\infty}^{m} a_i x^i$ and $g(x) = \sum_{j=-\infty}^{n} b_j x^j \in R((x^{-1};\sigma,\delta))$ with $f(x)g(x) = 0$. Then $(ef(x))(eg(x)) = 0$ and $((1-e)f(x))((1-e)g(x)) = 0$. Since eR and $(1-e)R$ are (σ,δ)-SILS Armendariz, we have $ea_i x^i eb_j x^j = 0$ and $(1-e)a_i x^i (1-e)b_j x^j = 0$, for each i,j. On the other hand, since $\sigma(e) = e$ and $\delta(e) = 0$, then we have $\sigma(eb_j) = e\sigma(b_j)$ and $\delta(eb_j) = e\delta(b_j)$. Hence, one can see that $ea_i x^i eb_j x^j = e(a_i x^i b_j x^j) = 0$ and $(1-e)a_i x^i(1-e)b_j x^j = (1-e)(a_i x^i b_j x^j) = 0$. Therefore $a_i x^i b_j x^j = e(a_i x^i b_j x^j) + (1-e)(a_i x^i b_j x^j) = 0$. Hence R is (σ,δ)-SILS Armendariz.

Let R be a ring and σ denotes an endomorphism of R with $\sigma(1) = 1$. In [22] the authors introduced *skew triangular matrix* ring as a set of all triangular matrices with addition point-wise and a new multiplication subject to the condition $E_{ij}r = \sigma^{j-i}(r)E_{ij}$. So $(a_{ij})(b_{ij}) = (c_{ij})$, where $c_{ij} = a_{ii}b_{ij} + a_{i,i+1}\sigma(b_{i+1,j}) + \cdots + a_{ij}\sigma^{j-i}(b_{jj})$, for each $i \leq j$ and denoted it by $T_n(R, \sigma)$.

The subring of the skew triangular matrices with constant main diagonal is denoted by $S(R, n, \sigma)$; and the subring of the skew triangular matrices with constant diagonals is denoted by $T(R, n, \sigma)$. We can denote $A = (a_{ij}) \in T(R, n, \sigma)$ by $(a_{11}, \ldots, a_{1n})$. Then $T(R, n, \sigma)$ is a ring with addition point-wise and multiplication given by:
$(a_0, \ldots, a_{n-1})(b_0, \ldots, b_{n-1}) = (a_0b_0, a_0 * b_1 + a_1 * b_0, \ldots, a_0 * b_{n-1} + \cdots + a_{n-1} * b_0)$, with $a_i * b_j = a_i\sigma^i(b_j)$, for each i and j. Therefore, clearly one can see that $T(R, n, \sigma) \cong R[x; \sigma]/(x^n)$, where (x^n) is the ideal generated by x^n in $R[x; \sigma]$. Also we consider the following two subrings of $S(R, n, \sigma)$, as follows:

$$A(R, n, \sigma) = \left\{ \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} \sum_{i=1}^{n-j+1} a_j E_{i,i+j-1} + \sum_{j=\lfloor \frac{n}{2} \rfloor + 1}^{n} \sum_{i=1}^{n-j+1} a_{i,i+j-1} E_{i,i+j-1} \right\};$$

$$B(R, n, \sigma) = \{A + rE_{1k} \mid A \in A(R, n, \sigma) \text{ and } r \in R\} \qquad n = 2k \geq 4.$$

Let α be an endomorphism of a ring R, σ an automorphism of R and δ an σ-derivation of R such that $\alpha\sigma = \sigma\alpha$ and $\delta\alpha = \alpha\delta$. The automorphism σ of R is extended to the automorphism $\bar{\sigma} : S \to S$ defined by $\bar{\sigma}((a_{ij})) = (\sigma(a_{ij}))$ and the σ-derivation δ of R is also extended to $\bar{\delta} : S \to S$ defined by $\bar{\delta}((a_{ij})) = (\delta(a_{ij}))$, where S is one of the rings $S(R, n, \alpha)$, $A(R, n, \alpha)$, $B(R, n, \alpha)$ or $T(R, n, \alpha)$. Also, the map $\bar{\alpha} : R((x^{-1}; \sigma, \delta)) \to R((x^{-1}; \sigma, \delta))$ defined by $\bar{\alpha}(\sum_{i=-\infty}^{m} a_i x^i) = \sum_{i=-\infty}^{m} \alpha(a_i)x^i$ is an endomorphism of $R((x^{-1}; \sigma, \delta))$.

Proposition 2.18. *Let σ be a rigid automorphism and α an endomorphism of a ring R such that $\alpha\sigma = \sigma\alpha$. If R is an α-rigid ring, then $R((x^{-1}; \sigma, \delta))$ is $\bar{\alpha}$-rigid.*

Proof. Let $f(x) = \sum_{i=-\infty}^{m} a_i x^i \in R((x^{-1}; \sigma, \delta))$ and $f(x)\bar{\alpha}(f(x)) = 0$. So we have $a_m\sigma^m(\alpha(a_m)) = 0$ and consequently $a_m\alpha(\sigma^m(a_m)) = 0$, since $\alpha\sigma = \sigma\alpha$. Thus $a_m\sigma^m(a_m) = 0$, since R is α-rigid and so $a_m = 0$, since R is σ-rigid. Hence $f(x) = 0$ and the proof is complete.

Theorem 2.19. *Let σ be an automorphism and δ be any σ-derivation of a ring R. Then R is σ-rigid if and only if R is reduced and (σ, δ)-SILS Armendariz.*

Proof. It is clear that each σ-rigid ring is reduced. Also, by Theorem 2.6 each σ-rigid ring is (σ, δ)-SILS Armendariz. Now, suppose that R is reduced and (σ, δ)-SILS Armendariz. We will prove that R is σ-rigid. Let $a\sigma(a) = 0$ for $a \in R$. Now, consider the elements $f(x) = \sigma(a) + \sigma(a)x$ and $g(x) = a - \sigma(a)x \in R((x^{-1}; \sigma, \delta))$, then it is not hard to see that $f(x)g(x) = 0$. Since R is (σ, δ)-SILS Armendariz, we have $\sigma(a)\sigma(a) = 0$ and hence $a = 0$, since σ is automorphism and R is reduced.

Theorem 2.20. *Let α be a rigid endomorphism of a ring R. Then the following statements are equivalent*:

(i) *R is an σ-rigid ring*;

(ii) *For each integer $n \geq 2$, D is an $(\bar{\sigma}, \bar{\delta})$-SILS Armendariz ring*;

(iii) *For some integer $n \geq 2$, D is an $(\bar{\sigma}, \bar{\delta})$-SILS Armendariz ring,*

where D is one of the rings $A(R,n,\alpha)$, $B(R,n,\alpha)$ or $T(R,n,\alpha)$.

Proof. (i) $\Rightarrow$ (ii) We only prove this theorem for the case $D = T(R,n,\alpha)$, because the proof of the other cases are similar. It is not hard to see that there exists an isomorphism of rings $\varphi : T(R,n,\alpha)((x^{-1};\bar{\sigma},\bar{\delta})) \to T\left(R((x^{-1};\sigma,\delta)),n,\bar{\alpha}\right)$, given by $\varphi(\sum_{k=-\infty}^{r} A_k x^k) = (f_{ij})$, where $A_k = (a_{ij}^{(k)})$ in $T(R,n,\alpha)$ and $f_{ij}(x) = \sum_{k=-\infty}^{r} a_{ij}^{(k)} x^k$ in $R((x^{-1};\sigma,\delta))$, for each $k \leq r$ and $1 \leq i,j \leq n$. Let $p(x) = \sum_{k=-\infty}^{r} A_k x^k$ and $q(x) = \sum_{l=-\infty}^{s} B_l x^l$ be elements in $T(R,n,\alpha)((x^{-1};\bar{\sigma},\bar{\delta}))$ such that $p(x)q(x) = 0$, where $A_k = (a_{ij}^{(k)})$ and $B_l = (b_{ij}^{(l)})$ in $T(R,n,\alpha)$, for $k \leq r$ and $l \leq s$. Thus $(f_{ij}(x))(g_{ij}(x)) = 0$, where $f_{ij}(x) = \sum_{k=-\infty}^{r} a_{ij}^{(k)} x^k$ and $g_{ij}(x) = \sum_{l=-\infty}^{s} b_{ij}^{(l)} x^l$ in $R((x^{-1};\sigma,\delta))$, for $1 \leq i,j \leq n$. Since R is α-rigid, $R((x^{-1};\sigma,\delta))$ is $\bar{\alpha}$-rigid, by Proposition 2.18. Thus $f_{iw}(x)g_{wj}(x) = 0$, for each $i,j,w \in \{1,\ldots,n\}$. Also, R is an (σ,δ)-SILS Armendariz ring, by Theorem 2.6, since R is σ-rigid. So $a_{iw}^{(k)} x^k b_{wj}^{(l)} x^l = 0$, for each $k \leq r$ and $l \leq s$. Thus $A_k x^k B_l x^l = 0$ and hence $T(R,n,\alpha)$ is an $(\bar{\sigma},\bar{\delta})$-SILS Armendariz ring.

(ii) $\Rightarrow$ (iii) It is clear.

(iii) $\Rightarrow$ (i) Assume that for some integer $n \geq 2$, $T(R,n,\alpha)$ is an $(\bar{\sigma},\bar{\delta})$-SILS Armendariz. First of all notice that R is (σ,δ)-SILS Armendariz. This is because, if we take $f(x) = \sum_{i=-\infty}^{m} a_i x^i$ and $g(x) = \sum_{j=-\infty}^{n} b_j x^j \in R((x^{-1};\sigma,\delta))$ such that $f(x)g(x) = 0$, then we have $F(x)G(x) = 0$, where $F(x) = \sum_{i=-\infty}^{m} A_i x^i$ and $G(x) = \sum_{j=-\infty}^{n} B_j x^j \in T(R,n,\alpha)((x^{-1};\bar{\sigma},\bar{\delta}))$, such that $A_i = a_i I_n$ and $B_j = b_j I_n$, for each i and j. Thus $A_i x^i B_j x^j = 0$ and consequently $a_i x^i b_j x^j = 0$. Hence R is (σ,δ)-SILS Armendariz, and hence $\sigma(e) = e$ and $\delta(e) = 0$ for each idempotent $e \in R$, by Proposition 2.9. Now, we shall prove that R is σ-rigid. Let $n = 2$ and $r\sigma(r) = 0$. Suppose that $f(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} \alpha\sigma(r) & 0 \\ 0 & \alpha\sigma(r) \end{pmatrix} x$ and $g(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \sigma(r) & 0 \\ 0 & \sigma(r) \end{pmatrix} x \in T(R,2,\alpha)((x^{-1};\bar{\sigma},\bar{\sigma}))$. Then it is not hard to see that $f(x)g(x) = 0$, since $r\sigma(r) = 0$ and α is compatible. So we have $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \sigma(r) & 0 \\ 0 & \sigma(r) \end{pmatrix} = 0$, since $T(R,2,\alpha)$ is $(\bar{\sigma},\bar{\delta})$-SILS Armendariz. Now, since α is compatible and σ is an automorphism, we have $r = 0$, as needed. Next, suppose that there exist an integer $n \geq 3$ such that $T(R,n,\alpha)$ is $(\bar{\sigma},\bar{\delta})$-SILS Armendariz and also let $r\sigma(r) = 0$ for $r \in R$. Taking the elements $p(x) = (0,0,1,0,\ldots,0) - (0,\alpha\sigma(r),0,\ldots,0)x$ and $q(x) = (0,0,\ldots,0,1,0) - (0,\ldots,0,\sigma(r),0,0)x$ in $T(R,n,\alpha)((x^{-1};\bar{\sigma},\bar{\sigma}))$, by a similar argument as above we can see that $p(x)q(x) = 0$, and hence $(0,0,1,0,\ldots,0)(0,\ldots,0,\sigma(r),0,0) = 0$, since $T(R,n,\alpha)$ is $(\bar{\sigma},\bar{\delta})$-SILS Armendariz. Then $\alpha(\sigma(r)) = 0$ and hence $r = 0$, since α is compatible and σ is an automorphism of R.

Recall that $A(R,2) = T(R,R)$ is the trivial extension of R. So by Theorem 2.20, if R is an σ-rigid ring, then $T(R,R)$ is $(\bar{\sigma},\bar{\delta})$-SILS Armendariz ring.
But $S(R,n,\alpha)$ is not $(\bar{\sigma},\bar{\delta})$-SILS Armendariz ring, where $n > 3$, even R is an σ-rigid ring. Since $E_{12}E_{34} = 0$ and $E_{12}E_{23}E_{34} \neq 0$. Thus by Theorem 2.11, $S(R,n,\alpha)$ is not $(\bar{\sigma},\bar{\delta})$-SILS Armendariz ring, for $n > 3$.

In the following, we study the (σ,δ)-SILS Armendariz property of classical quotient rings over (σ,δ)-SILS Armendariz rings. A ring R is called *right Ore* if given $a,b \in R$ with b regular (elements that are neither left nor right zero-divisors), there exist $a_1, b_1 \in R$ with b_1 regular such that $ab_1 = ba_1$. Left Ore rings can be defined similarly. It is well-known that R is a right Ore ring if and only if the classical right quotient ring of R exists. If both right and left quotient rings exist, then they are equal. Let F be a field and R the free algebra in two indeterminates over F. Then R is a domain but can not be right (left) Ore. It is also well-known that R is a right Ore domain if and only if the classical right quotient ring of R is a division ring. Let R be a ring with a classical right quotient ring Q and let $C(R)$ denotes the set of all regular elements of R. Then each automorphism σ and each σ-derivation δ of R, extends to Q, respectively, by setting $\bar{\sigma}(rc^{-1}) = \sigma(r)\sigma(c)^{-1}$ and $\bar{\delta}(rc^{-1}) = (\delta(r) - \sigma(r)\sigma(c)^{-1}\delta(c))c^{-1}$, for each $r \in R$ and $c \in C(R)$.

Theorem 2.21. *Let R be an Ore ring with an automorphism σ and σ-derivation δ. Then R is* (*linearly*) *(σ,δ)-SILS Armendariz if and only if the classical quotient ring Q of R is* (*linearly*) *$(\bar{\sigma},\bar{\delta})$-SILS Armendariz.*

Proof. We only need to prove the sufficient condition. First we claim that for each element $g(x) \in Q((x^{-1};\bar{\sigma},\bar{\delta}))$, there exists $c \in C(R)$ such that $g(x) = f(x)c^{-1}$, for some $f(x) \in R((x^{-1};\sigma,\delta))$, or equivalently $g(x)c \in R((x^{-1};\sigma,\delta))$. For each $g(x) = \sum_{j=-k}^{n} b_j x^j \in Q((x^{-1};\bar{\sigma},\bar{\delta}))$ (all b_j are non-zero and $n \in \mathbb{Z}$), we define $length(g(x)) = k + n + 1$, and we work by induction on $length(g(x))$ to prove our claim. If $length(g(x)) = 1$, then $g(x) = bc_1^{-1}x^{-k}$, where $c_1 \in C(R)$. Taking $c = \sigma^k(c_1)$, we deduce that $g(x) = (bx^{-k})c^{-1}$ and $c \in C(R)$, since σ is an automorphism. Now assume that the claim is true for all $g(x) \in Q((x^{-1};\bar{\sigma},\bar{\delta}))$ with $length(g(x)) < l$ and let $g(x) = \sum_{j=-k}^{n} b_j x^j \in Q((x^{-1};\bar{\sigma},\bar{\delta}))$ with $n + k = l - 1$, where all $b_j = a_j c_j^{-1}$ are non-zero. Let $c_n = \sigma^n(d)$, for some $d \in C(R)$. Then $a_n c_n^{-1} x^n d = a_n x^n$. So we have $g(x)d = (a_{-k}c_{-k}^{-1}x^{-k} + \cdots + a_{n-1}c_{n-1}^{-1}x^{n-1})d + a_n x^n$. By induction hypothesis, there exists $e \in C(R)$ such that $(a_{-k}c_{-k}^{-1}x^{-k} + \cdots + a_{n-1}c_{n-1}^{-1}x^{n-1})e \in R((x^{-1};\sigma,\delta))$. Thus we have $g(x)de = (a_{-k}c_{-k}^{-1}x^{-k} + \cdots + a_{n-1}c_{n-1}^{-1}x^{n-1})de + a_n x^n e \in R((x^{-1};\sigma,\delta))$, where $de \in C(R)$, and the result follows. Now, suppose that R is (σ,δ)-SILS Armendariz and let $f(x) = \sum_{i=-\infty}^{m} a_i c_i^{-1} x^i$ and $g(x) = \sum_{j=-\infty}^{n} b_j d_j^{-1} x^j \in Q((x^{-1};\bar{\sigma},\bar{\delta}))$ such that $f(x)g(x) = 0$. Let $a_i c_i^{-1} = p^{-1}a_i'$ and $b_j d_j^{-1} = q^{-1}b_j'$ with $p, q \in C(R)$. Then $(\sum_{i=-\infty}^{m} a_i' x^i)q^{-1}(\sum_{j=-\infty}^{n} b_j' x^j) = 0$. By above claim, there exists $s \in C(R)$ and $\sum_{j=-\infty}^{n} b_j'' x^j \in R((x^{-1};\sigma,\delta))$, such that $q^{-1}(\sum_{j=-\infty}^{n} b_j' x^j) = (\sum_{j=-\infty}^{n} b_j'' x^j)s^{-1}$. Hence $(\sum_{i=-\infty}^{m} a_i' x^i)(\sum_{j=-\infty}^{n} b_j'' x^j) = 0$ in $R((x^{-1};\sigma,\delta))$. Since R is (σ,δ)-SILS Armendariz, we have $a_i' x^i b_j'' x^j = 0$, for each $i \leq m$ and each $j \leq n$. Therefore $p^{-1}a_i' x^i q^{-1} b_j' x^j = 0$. Hence $(a_i c_i^{-1})x^i(b_j d_j^{-1})x^j = 0$, for each $i \leq m$ and $j \leq n$.

3. Radicals of Skew Inverse Laurent Series Rings

In the theory of rings, it is an important issue to investigate the coincidence of certain radicals on a given class of rings. Perhaps the greatest unsolved problem in non-commutative ring theory today is the Köthe's Conjecture, which posits that a ring with no non-zero nil ideals has no non-zero nil one-sided ideals. The Köthe Conjecture has been resolved in several special cases, including for rings with Krull dimension, for PI rings, and for algebras over uncountable fields. We will presently add σ-compatible (σ,δ)-SILS Armendariz rings to this list. For more information about the behavior of radical properties under polynomial extensions, we refer the reader to the recent book [15].

Following [29], for a ring R, let $N(R)$ denote the set of nilpotent elements of R, $N_0(R)$ the Wedderburn radical of R (that is, the sum of all nilpotent ideals of R), $Ni\ell_*(R)$ the lower nil radical of R (i.e., the prime radical of R), L-$rad(R)$ the Levitzki radical of R (i.e., sum of all locally nilpotent ideals of R), $Ni\ell^*(R)$ the upper nilradical of R (i.e., sum of all nil ideals of R), and $\mathrm{A}(R)$ the sum of all nil left ideals of R (which coincides with the sum of all nil right ideals of R). The Köthe Conjecture is equivalent to the statement that $\mathrm{A}(R)$ is always nil, that is, $Ni\ell^*(R) = A(R)$ for every ring R.

Theorem 3.1. *If R is σ-compatible (σ,δ)-SILS Armendariz ring, then we have*:

$$N_0(R) = Ni\ell_*(R) = L\text{-}rad(R) = Ni\ell^*(R) = A(R).$$

Proof. It is enough to prove that $A(R) \subseteq N_0(R)$. Now let $x \in A(R)$. Since $A(R)$ is an ideal of R and $A(R) \subseteq N(R)$, it follows that $RxR \subseteq N(R)$. Then $x^n = 0$ for some $n \in \mathbb{N}$. Since $x^{n-1}x = 0$ and RxR is a nil ideal, we have $x^{n-1}yx = 0$, for each $y \in RxR$, by Theorem 2.11. Thus $x^{n-1}(RxR)x = 0$. Hence $x^{n-2}xRxRx = 0$. By continuing this method, we have $x(RxR)x(RxR)x \cdots x(RxR)x = 0$. Therefore we get $(RxR)^{2n-1} = 0$ and we are done.

Corollary 3.2. (i) *Each semiprime σ-compatible (σ,δ)-SILS Armendariz ring has no non-zero nil one-sided ideals.*
(ii) *Each σ-compatible (σ,δ)-SILS Armendariz ring satisfies the Köthe's Conjecture.*

Proposition 3.3. *Let R be a σ-compatible (σ,δ)-SILS Armendariz ring and $f_i \in R((x^{-1};\sigma,\delta))$, for $1 \leq i \leq n$. If $f_1 \cdots f_n = 0$, then $a_1a_2 \cdots a_n = 0$, where a_i is any coefficient of f_i for each i.*

Proof. We work by induction on n. By Theorem 2.10, the result is true for $n = 2$. Now assume that the result is true for all $m < n$, and let $f_1f_2 \ldots f_n = 0$. Then by Theorem 2.10, $a_1a_h = 0$ where $a_1 \in C_{f_1}$ and $a_h \in C_{f_h}$, where $h = f_2f_3 \ldots f_n$. So $(a_1f_2)f_3 \ldots f_n = 0$. But a_1f_2 is an element of $R((x^{-1};\sigma,\delta))$ with coefficient a_1b_j, where b_j is a coefficient of f_2. By the induction hypothesis, for each coefficient a_2 of f_2 and each coefficient a_i of f_i, with $3 \leq i \leq n$, we have $a_1a_2 \cdots a_n = 0$ and the proof is complete.

Lemma 3.4. *Let R be a σ-compatible (σ,δ)-SILS Armendariz ring. Then we have*:
(i) *If $a_1a_2\cdots a_n=0$, then we have $\sigma^{k_1}(a_1)\sigma^{k_2}(a_2)\cdots\sigma^{k_n}(a_n)=0$, for all integers $k_1,k_2,\ldots,k_n$.*
(ii) *If $\sigma^{k_1}(a_1)\sigma^{k_2}(a_2)\cdots\sigma^{k_n}(a_n)=0$, for some integers $k_1,k_2,\ldots,k_n$, then we have $a_1a_2\cdots a_n=0$.*
(iii) *If $a_1a_2\cdots a_n=0$, then we have $\delta^{k_1}(a_1)\delta^{k_2}(a_2)\cdots\delta^{k_n}(a_n)=0$, for all positive integers $k_1,k_2,\ldots,k_n$.*

Proof. (i) Notice that if $ab=0$, then $a\sigma^k(b)=\sigma^k(a)b=0$ and also $\sigma^k(a)\delta^l(b)=\delta^l(a)\sigma^k(b)=0$ for all integer k and any positive integer l. Also, if $\sigma^k(a)b=0$ for some integer k, then $ab=0$. Now, we work by induction on n. Let $n=2$ and $a_1a_2=0$, then $\sigma^{k_1}(a_1)a_2=0$. Then we have $\sigma^{k_2}(\sigma^{k_1}(a_1)a_2)=0$ and so $\sigma^{k_1+k_2}(a_1)\sigma^{k_2}(a_2)=0$. So we have $a_1\sigma^{k_2}(a_2)=0$. Therefore $\sigma^{k_1}(a_1)\sigma^{k_2}(a_2)=0$. Now, suppose by induction that $\sigma^{k_1}(a_1)\sigma^{k_2}(a_2)\cdots\sigma^{k_l}(a_l)=0$, for all $l<n$ and all positive integers $k_1,k_2,\ldots,k_l$, when $a_1a_2\cdots a_l=0$. Now, let $a_1a_2\cdots a_n=0$. Thus $a_1\ldots a_{n-2}(a_{n-1}a_n)=0$. Then $\sigma^{k_1}(a_1)\cdots\sigma^{k_{n-2}}(a_{n-2})\sigma^{k_{n-1}}(a_{n-1}a_n)=0$, by induction hypothesis. So $\sigma^{k_1}(a_1)\cdots\sigma^{k_{n-2}}(a_{n-2})\sigma^{k_{n-1}}(a_{n-1})\sigma^{k_{n-1}}(a_n)=0$. Using σ-compatibility of R, we have $\sigma^{k_1}(a_1)\cdots\sigma^{k_{n-2}}(a_{n-2})\sigma^{k_{n-1}}(a_{n-1})a_n=0$, and then $\sigma^{k_1}(a_1)\cdots\sigma^{k_{n-2}}(a_{n-2})\sigma^{k_{n-1}}(a_{n-1})\sigma^{k_n}(a_n)=0$. This finishes our proof.
(ii) The proof is similar to that of (i).
(iii) Since $a_1a_2\cdots a_n=0$, then $\delta^{k_1}(a_1)a_2\cdots a_n=0$. So by Proposition 2.5(1) $\delta^{k_1}(a_1)\delta(a_2\cdots a_n)=0$, and hence $\delta^{k_1}(a_1)\delta(a_2)a_3\cdots a_n+\delta^{k_1}(a_1)\sigma(a_2)\delta(a_3\cdots a_n)=0$. Since $\delta^{k_1}(a_1)a_2\cdots a_n=0$, then we have $\sigma(\delta^{k_1}(a_1)a_2)\delta(a_3\cdots a_n)=0$ and then we have $\delta^{k_1}(a_1)\sigma(a_2)\delta(a_3\cdots a_n)=0$. So $\delta^{k_1}(a_1)\delta(a_2)a_3\cdots a_n=0$ and then $\delta^{k_1}(a_1)\delta(\delta(a_2)a_3\cdots a_n)=0$, by Proposition 2.5(1). Hence $\delta^{k_1}(a_1)\delta^2(a_2)(a_3\cdots a_n)+\delta^{k_1}(a_1)\sigma(\delta(a_2))\delta(a_3\cdots a_n)=0$. Since we have $\delta^{k_1}(a_1)\delta(a_2)a_3\cdots a_n=0$, then $\sigma(\delta^{k_1}(a_1)\delta(a_2))\delta(a_3\cdots a_n)=0$, and hence we have $\delta^{k_1}(a_1)\sigma(\delta(a_2))\delta(a_3\cdots a_n)=0$. Then we obtain that $\delta^{k_1}(a_1)\delta^2(a_2)(a_3\cdots a_n)$. Continuing in this process we get $\delta^{k_1}(a_1)\delta^{k_2}(a_2)(a_3\cdots a_n)=0$. So $\delta^{k_1}(a_1)\delta^{k_2}(a_2)\delta(a_3\cdots a_n)=0$ and hence $\delta^{k_1}(a_1)\delta^{k_2}(a_2)\delta(a_3)(a_4\cdots a_n)+\delta^{k_1}(a_1)\delta^{k_2}(a_2)\sigma(a_3)\delta(a_4\cdots a_n)=0$, by Proposition 2.5(1). Since we proved that $\delta^{k_1}(a_1)\delta^{k_2}(a_2)(a_3\cdots a_n)=0$, then we have $\sigma(\delta^{k_1}(a_1)\delta^{k_2}(a_2)a_3)\delta(a_4\cdots a_n)=0$, and hence $\delta^{k_1}(a_1)\delta^{k_2}(a_2)\sigma(a_3)\delta(a_4\cdots a_n)=0$. Then $\delta^{k_1}(a_1)\delta^{k_2}(a_2)\delta(a_3)(a_4\cdots a_n)=0$. By continuing in this fashion we receive that $\delta^{k_1}(a_1)\delta^{k_2}(a_2)\delta^{k_3}(a_3)(a_4\cdots a_n)=0$ and by a similar way as above, for all positive integers $k_1,k_2,\ldots,k_n$, we get $\delta^{k_1}(a_1)\delta^{k_2}(a_2)\delta^{k_3}(a_3)\cdots\delta^{k_n}(a_n)=0$, as desired.

Proposition 3.5. *Let R be a σ-compatible (σ,δ)-SILS Armendariz ring. If $f(x)$ and $g(x)$ be in $R((x^{-1};\sigma,\delta))$ and $h(x)\in N\left(R((x^{-1};\sigma,\delta))\right)$, then $f(x)g(x)=0$ implies $f(x)h(x)g(x)=0$.*

Proof. Let $f(x)=\sum_{i=-\infty}^{m}a_ix^i$, $g(x)=\sum_{j=-\infty}^{n}b_jx^j$ and $h(x)=\sum_{k=-\infty}^{l}c_kx^k\in R((x^{-1};\sigma,\delta))$ such that for some positive integer p, we have $[h(x)]^p=0$. Then by Proposition 3.3, $c_k^p=0$ for each $k\leq l$. On the other hand, since $f(x)g(x)=0$, then by Proposition 3.3, $a_ib_j=0$ for each $i\leq m$ and $j\leq n$. Hence by Theorem 2.11, $a_ic_kb_j=0$ for each $i\leq m$, $j\leq n$ and $k\leq l$. Then by Lemma 3.4, $\sigma^{r_i}(a_i)\sigma^{s_k}(c_k)\sigma^{t_j}(b_j)=0$ for all integers r_i,s_k,t_j and also $\delta^{m_i}(a_i)\delta^{n_k}(c_k)\delta^{q_j}(b_j)=0$, for all positive integers m_i,n_k,q_j. So $f(x)h(x)g(x)=0$, and the result follows.

Theorem 3.6. *Let R be σ-compatible (σ,δ)-SILS Armendariz ring. Then we have:*

$$N_0(R((x^{-1};\sigma,\delta))) = Ni\ell_*(R((x^{-1};\sigma,\delta))) = L\text{-}rad(R((x^{-1};\sigma,\delta))) =$$
$$Ni\ell^*(R((x^{-1};\sigma,\delta))) = A(R((x^{-1};\sigma,\delta))).$$

Proof. Using Proposition 3.5, the proof is similar to that of Theorem 3.1.

Corollary 3.7. *Let R be σ-compatible (σ,δ)-SILS Armendariz ring. Then the skew inverse Laurent series ring $R((x^{-1};\sigma,\delta))$ satisfies the Köthe's Conjecture.*

Theorem 3.8. *Let R be a σ-compatible (σ,δ)-SILS Armendariz ring and let $S = R((x^{-1};\sigma,\delta))$. Then we have $Ni\ell^*(S)\cap R = N_0(R)$.*

Proof. Let $a \in Ni\ell^*(S)\cap R$. Clearly, $a \in Ni\ell^*(R)$. Thus by Theorem 3.1, we have $a \in N_0(R)$. Hence $Ni\ell^*(S)\cap R \subseteq N_0(R)$. Conversely, let $a \in N_0(R)$. Therefore $(RaR)^k = 0$, for some $k \geq 1$. We prove that $(f(x)ag(x))^k = 0$, where $f(x) = \sum_{i=-\infty}^{m} r_i x^i$ and $g(x) = \sum_{j=-\infty}^{n} s_j x^j \in S$. Since $(RaR)^k = 0$, we have $\prod_{l=1}^{k} b_l a c_l = 0$, where b_l and c_l are the coefficients of $f(x)$ and $g(x)$, respectively, for all $1 \leq l \leq k$. So $(f(x)ag(x))^k = 0$, by Lemma 3.4. Hence $a \in Ni\ell^*(S)$ and so $N_0(R) \subseteq Ni\ell^*(S)\cap R$.

Corollary 3.9. *Let R be a σ-compatible (σ,δ)-SILS Armendariz ring and let $S = R((x^{-1};\sigma,\delta))$. Then we have:*

$N_0(R) = N_0(S)\cap R = Ni\ell_*(R) = Ni\ell_*(S)\cap R = L\text{-}rad(R) = L\text{-}rad(S)\cap R = Ni\ell^*(R) = Ni\ell^*(S)\cap R = A(R) = A(S)\cap R$.

Proof. The result follows by Theorems 3.1, 3.6 and 3.8.

Corollary 3.10. *Let R be a σ-compatible (σ,δ)-SILS Armendariz ring and let $S = R((x^{-1};\sigma,\delta))$. Then:*

(i) $J(R[x]) = N_0(R)[x] = \mathrm{Nil}_*(R)[x]$=L-$rad(R)[x]$=$\mathrm{Nil}^*(R)[x]$ =A$(R)[x]$.

(ii) $J(S[y]) = N_0(S)[y] = \mathrm{Nil}_*(S)[y]$=L-$rad(S)[y]$=$\mathrm{Nil}^*(S)[y]$ =A$(S)[y]$.

Proof. It follows from Theorems 3.1 and 3.6 and [29, Exercise 10.25].

Corollary 3.11. *Let R be a σ-compatible (σ,δ)-SILS Armendariz ring and let $S = R((x^{-1};\sigma,\delta))$. Then:*

(i) $\mathrm{Nil}^*(M_n(R))=M_n(\mathrm{Nil}^*(R))=M_n(\mathrm{Nil}_*(R))=\mathrm{Nil}_*(M_n(R))$.

(ii) $\mathrm{Nil}^*(M_n(S))=M_n(\mathrm{Nil}^*(S))=M_n(\mathrm{Nil}_*(S))=\mathrm{Nil}_*(M_n(S))$.

Proof. It follows from Theorems 3.1 and 3.6 and [29, Exercise 10.25].

Theorem 3.12. *Let R be a σ-compatible (σ,δ)-SILS Armendariz ring and let $S = R((x^{-1};\sigma,\delta))$. If R satisfies any one of the following conditions, then we have $Ni\ell^*(S) = N_0(R)((x^{-1};\sigma,\delta))$.*

(i) *R is left or right Goldie*;
(ii) *R has the ACC on ideals*;
(iii) *R is a ring with right Krull dimension*;
(iv) *R has the ACC and DCC on left annihilators*;
(v) *R is a ring with ACC on both right and left annihilators.*

Proof. First, we prove that if R is σ-compatible (σ,δ)-SILS Armendariz ring, then $Ni\ell^*(S) \subseteq N_0(R)((x^{-1};\sigma,\delta))$. Let $f(x) = \sum_{i=-\infty}^{m} a_i x^i \in Ni\ell^*(S)$. Thus the ideal generated by $f(x)$ in S is nil. So Ra_iR is nil ideal in R and hence $a_i \in Ni\ell^*(R) = N_0(R)$, for each $i \leq m$, by Lemma 3.4. Conversely, let $f(x) = \sum_{i=-\infty}^{m} a_i x^i \in N_0(R)((x^{-1};\sigma,\delta))$. Since R satisfies one of the above conditions, $N_0(R)$ is a nilpotent ideal of R, by [30, Theorem 1], [29, Theorem 4.12], [32], [24, Theorem 1] and [10, Theorem 1.34], respectively. Therefore $(N_0(R))^k = 0$, for some $k \geq 1$. Thus, by Lemma 3.4, we have $[g(x)f(x)h(x)]^k = 0$. Hence $N_0(R)((x^{-1};\sigma,\delta)) \subseteq Ni\ell^*(S)$.

A ring R is called 2-*primal* if its prime radical contains every nilpotent element of R. G.F. Birkenmeier et al. [9, Proposition 2.6], proved that the 2-primal condition is inherited by ordinary polynomial extensions. Ore extensions do not generally preserve the 2-primal condition (see [13] and [14]).

Theorem 3.13. *Let R be a σ-compatible (σ,δ)-SILS Armendariz ring and let $S = R((x^{-1};\sigma,\delta))$. Then we have*:
(1) *R is 2-primal if and only if S is 2-primal.*
(2) *R is semi-commutative if and only if S is semi-commutative.*
(3) *R is reversible if and only if S is reversible.*
(4) *R is symmetric if and only if S is symmetric.*

Proof. (1) First suppose that S is 2-primal. Let $a \in Ni\ell(R)$. So $a \in Ni\ell(S)$ and $a \in Ni\ell_*(S) \cap R$. Thus $a \in Ni\ell_*(R)$, by Corollary 3.9. Hence R is 2-primal. Conversely, suppose that R is 2-primal. Let $f(x) = \sum_{i=-\infty}^{m} a_i x^i \in Ni\ell(S)$. So $[f(x)]^k = 0$, for some $k \geq 1$. Therefore $\prod_{l=1}^{k} r_l = 0$, where r_l is a coefficient of $f(x)$, for each $1 \leq l \leq k$, by Proposition 3.3. So $a_i \in Ni\ell(R) = Ni\ell_*(R) = Ni\ell^*(R)$, for all i. Therefore Ra_iR is nil. Now, we get $r_1(r_2 \cdots r_k) = 0$. Thus $r_1(Rs_1R)r_2 \cdots r_k = 0$, where s_1 is a coefficient of $f(x)$, by Theorem 2.11. By continuing in this way, we get $r_1Rs_1Rr_2Rs_2 \cdots s_{k-1}Rr_k = 0$, where s_i is the coefficient of $f(x)$, for each $1 \leq i \leq k-1$. Therefore $(g(x)f(x)h(x))^{2k-1} = 0$, for each $g(x)$ and $h(x) \in S$, by Lemma 3.4. Hence $f(x) \in Ni\ell^*(S) = Ni\ell_*(S)$.
(2) Let R be a semi-commutative ring and $f(x)g(x) = 0$, where $f(x) = \sum_{i=-\infty}^{m} a_i x^i$ and $g(x) = \sum_{j=-\infty}^{n} b_j x^j \in S$. Then $a_ib_j = 0$ for each $i \leq m$ and $j \leq n$, by Theorem 2.10. Since R is semi-commutative, $a_irb_j = 0$ for each $r \in R$, $i \leq m$ and $j \leq n$. Hence by Lemma 3.4, $\sigma^{k_1}(a_i)\sigma^{k_2}(r)\sigma^{k_3}(b_j) = 0$ and $\delta^{k_4}(a_i)\delta^{k_5}(r)\delta^{k_6}(b_j) = 0$ for each $i \leq m$, $j \leq n$ and integers k_i. Then $f(x)h(x)g(x) = 0$ for each $h(x) \in S$, and hence S is semi-commutative. The proof of (3) and (4) are similar to that of (2).

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