WHAT IS... Cyclic Sieving?

Victor Reiner, Dennis Stanton, and Dennis White

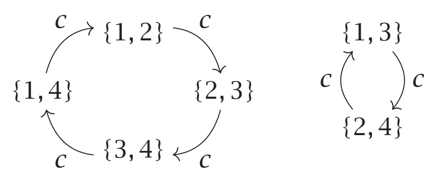
Many finite sets in combinatorics have both cyclic symmetry and a natural generating function. Surprisingly often the generating function evaluated at roots of unity counts symmetry classes. We call this the *cyclic sieving phenomenon*.

More precisely, let C be a cyclic group generated by an element c of order n acting on a finite set X. Given a polynomial $X(q)$ with integer coefficients in a variable q, we say that the triple $(X, X(q), C)$ *exhibits the cyclic sieving phenomenon (CSP)* if for all integers d, the number of elements fixed by c^d equals the evaluation $X(\zeta^d)$ where $\zeta = e^{\frac{2\pi i}{n}}$. In particular, $X(1)$ is the cardinality of X, so that $X(q)$ can be regarded as a *generating function* for X.

In the protoexample, X is the collection of all k-element subsets of $\{1, 2, \ldots, n\}$, and $X(q)$ is the renowned *q-binomial coefficient* or *Gaussian polynomial*

$$X(q) = \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]!_q}{[k]!_q[n-k]!_q} \qquad (1)$$

where $[m]!_q := [m]_q[m-1]_q \cdots [2]_q[1]_q$ and $[m]_q := 1 + q + q^2 + \cdots + q^{m-1}$. Let the generator c of C act by cycling the elements of a k-subset modulo n. One then finds [1, Theorem 1.1(b)] that this triple $(X, X(q), C)$ exhibits the CSP. For example, taking $n = 4$ and $k = 2$, one has c acting as shown here:



One can compute $X(q) = 1 + q + 2q^2 + q^3 + q^4$ from (1). Note that $X(1) = 6$, while $X((e^{\frac{2\pi i}{4}})^2) = X(-1) = 2$ counts the two subsets $\{\{1,3\},\{2,4\}\}$ fixed by c^2, and $X(e^{\frac{2\pi i}{4}}) = 0 = X((e^{\frac{2\pi i}{4}})^3)$ since no two-element subset is fixed by c or c^3.

The CSP was first defined in [1]. It has proven to be remarkably ubiquitous; see, for example, B. Sagan's excellent survey [3]. The special case of a CSP when C has order 2 was known as J. Stembridge's $q = -1$ *phenomenon* [4]. He gave interesting examples involving enumeration of *plane partitions* and *Young tableaux*.

Stembridge emphasized the value of a single q-formula $X(q)$ encompassing both the cardinality of X as $X(1)$ and a second enumeration $X(-1)$ of a symmetry class within X. A CSP triple $(X, X(q), C)$ generalizes his idea. The polynomial $X(q)$ packages as its n-th root of unity evaluations, or equivalently in its residue class modulo $q^n - 1$, all of the information about the cyclic action of C on X. In fact, given (X, C) there is always a unique (but generally *uninteresting*) choice of a polynomial $X(q)$ of degree at most $n - 1$ completing the triple, as the CSP is equivalent [1, Proposition 2.1(ii)] to the assertion that

$$X(q) \equiv \sum_{i=0}^{n-1} a_i q^i \bmod q^n - 1,$$

Victor Reiner and Dennis Stanton are professors of mathematics and Dennis White is professor emeritus of mathematics; all three are at the University of Minnesota. Their email addresses are, respectively, reiner@math.umn.edu, stanton@math.umn.edu, *and* white@math.umn.edu.

This work is partially supported by NSF grant DMS-1148634.

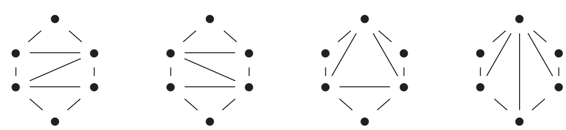
DOI: http://dx.doi.org/10.1090/noti1084

where a_i is the number of orbits of C on X in which the stabilizer cardinality divides i. Thus a CSP interprets combinatorially the coefficients of $X(q)$ when reduced mod $q^n - 1$; e.g., a_0 counts the total number of orbits on X, while a_1 counts the number of *free* orbits. Our protoexample with $n = 4$ and $k = 2$ has $X(q) \equiv 2 + q + 2q^2 + q^3 \bmod q^4 - 1$, so $a_0 = 2$ counts the two orbits in total, and $a_1 = 1$ counts the free orbit.

Here is a second example from [1]. Let X be the set of triangulations of a regular $(n+2)$-gon, with C a cyclic group of order $n+2$ rotating triangulations, and let

$$X(q) = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q,$$

a *q-Catalan number* considered by P. A. MacMahon. Then $(X, X(q), C)$ exhibits the CSP [1, Theorem 7.1]. For example, when $n = 4$, the four orbits of triangulations are represented by



while

$$\begin{aligned} X(q) &= \frac{1}{[5]_q} \begin{bmatrix} 8 \\ 4 \end{bmatrix}_q \\ &= 1 + q^2 + q^3 + 2q^4 + q^5 \\ &\quad + 2q^6 + q^7 + 2q^8 + q^9 + q^{10} + q^{12} \\ &\equiv 4 + q + 3q^2 + 2q^3 + 3q^4 + q^5 \bmod q^6 - 1, \end{aligned}$$

so that $a_0 = 4$ counts the four orbits, of which $a_1 = 1$ of them is free (the fourth orbit), while $a_2 = 3$ orbits (the first, second, fourth) have stabilizer size dividing 2, and $a_3 = 2$ orbits (the third, fourth) have stabilizer size dividing 3.

It was conjectured by the authors and verified by S.-P. Eu and T.-S. Fu that this triangulation example generalizes to a CSP triple $(X, X(q), C)$ in which X is the collection of *clusters* in a cluster algebra of finite type W à la S. Fomin and A. Zelevinsky, where C is generated by a *deformed Coxeter element* and $X(q)$ is a q-analogue of the *Catalan number* for W.

So what makes a generating function $X(q)$ "natural"? To some extent, this is in the eye of the beholder. Nevertheless, here are some conditions on $X(q)$ arising in many CSPs encountered so far:

(i) $X(q)$ is the *statistic generating function* for a map $s : X \to \{0, 1, 2, \ldots\}$; that is, $X(q) = \sum_{x \in X} q^{s(x)}$.

(ii) $X(q)$ has a simple *product formula.*

(iii) $X(q)$ at $q = p^d$ a prime power counts the points of a variety $X(\mathbb{F}_q)$ defined over the *finite field* $\mathbb{F}_q$.

(iv) $X(q^2) = \sum_i \beta_i q^i$ records the *Betti numbers* β_i of a complex variety $X(\mathbb{C})$.

(v) $X(q) = \sum_i \dim R_i \, q^i$ records the *Hilbert series* of some interesting graded ring $R = \bigoplus_i R_i$.

(vi) $X(q^2)$ is, up to a power of q, the *formal character* of an $SL_2(\mathbb{C})$-representation, that is, the sum $\sum_i \dim V_i \, q^i$ where V_i is the weight space on which a diagonal matrix with eigenvalues (q, q^{-1}) acts via the scalar q^i.

Our protoexample has each of these natural properties:

(a) After multiplying $X(q)$ by $q^{\binom{k+1}{2}}$, it is the statistic generating function for k-subsets A by their sum $s(A) = \sum_{a \in A} a$.

(b) The product formula for $X(q)$ is given in (1).

(c) $X(q)$ counts the points in the *Grassmannian of k-planes* in an n-dimensional vector space over $\mathbb{F}_q$.

(d) $X(q^2)$ records the Betti numbers for this Grassmannian over $\mathbb{C}$.

(e) When the symmetric group S_n permutes polynomials in n variables, $X(q)$ is the Hilbert series for the quotient ring[1] of the polynomials invariant under $S_k \times S_{n-k}$ after modding out the nonconstant polynomials invariant under S_n.

(f) $q^{-k(n-k)} X(q^2)$ is the formal character for the k-th exterior power of the n-dimensional $SL_2(\mathbb{C})$-irreducible.

In our triangulations example, the q-Catalan $X(q)$ has an interpretation as in (a), (b), (c) and a variation of (e). We know no interpretation like (d) or (f).

Some CSPs in the literature are proven via a *linear algebra paradigm* [1, §2]. Such proofs interpret $X(q)$ as in (d) or (e), giving a graded representation $V = \bigoplus_i V_i$ of the cyclic group C. One shows that $X(\zeta^d)$ equals the size of the c^d-fixed subset of X by computing the trace of c^d using two bases. The first basis is indexed by X and permuted by c, so that the trace of c^d is the size of the c^d-fixed subset. The second basis shows that c scales V_i by ζ^i, so that c^d has trace $X(\zeta^d)$.

A pleasing situation where this paradigm works generalizes (e) above. It arises from the *invariant theory* of finite subgroups W of $GL_n(\mathbb{C})$ generated by *reflections*, that is, elements whose fixed space is a complex hyperplane. T. Springer developed a theory of *regular elements* in such groups, which are the elements c that have an eigenvector fixed by none of the reflections of W. Using Springer's main result, one obtains [1, Theorem 8.2] a CSP triple from the coset space $X := W/W'$ for *any* subgroup W', with C generated by a regular element left-translating coset, and $X(q)$ is the quotient of the

[1] *This graded ring is isomorphic, after doubling degrees, to the cohomology of the Grassmannian in* (d).

Hilbert series for the W'-invariant polynomials over the Hilbert series for the W-invariant polynomials.

An intriguing CSP was conjectured by D. White involving *rectangular Young tableaux* and the cyclic action of *jeu-de-taquin promotion*. It has now seen several proofs via the linear algebra paradigm, first by B. Rhoades [2] and most recently by B. Fontaine and J. Kamnitzer. Such insightful proofs are rarer than we would like. Many known instances of CSPs, such as the triangulations example, have only been verified using a product formula for $X(q)$ to evaluate $X(\zeta^d)$ and comparing with known counts of symmetry classes.

We close with a perplexing example of this nature. Let X be the set of $n \times n$ *alternating sign matrices*: the matrices with $0, \pm 1$ entries whose row and column sums are all $+1$, and nonzero entries alternate in sign reading along any row or column. Here they are for $n = 3$:

$$\begin{array}{ccc} \begin{matrix}0&1&0\\1&0&0\\0&0&1\end{matrix} & \xrightarrow{c} & \begin{matrix}0&1&0\\0&0&1\\1&0&0\end{matrix} \\ {\scriptstyle c}\uparrow & & \downarrow{\scriptstyle c} \\ \begin{matrix}0&0&1\\1&0&0\\0&1&0\end{matrix} & \xleftarrow{c} & \begin{matrix}1&0&0\\0&0&1\\0&1&0\end{matrix} \end{array} \qquad \begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix} \underset{c}{\overset{c}{\rightleftarrows}} \begin{matrix}0&0&1\\0&1&0\\1&0&0\end{matrix} \qquad \begin{matrix}0&1&0\\1&-1&1\\0&1&0\end{matrix} \circlearrowleft c$$

Let C be the cyclic group of order 4 whose generator c rotates matrices through $90°$. Let

$$X(q) = \prod_{k=0}^{n-1} \frac{[3k+1]!_q}{[n+k]!_q}.$$

This triple $(X, X(q), C)$ exhibits the CSP, but we have no linear algebraic proof. Furthermore, $X(q)$ is only known as the generating function for *descending plane partitions* by weight and is not defined by a statistic on alternating sign matrices.

References

[1] V. Reiner, D. Stanton, and D. White, The cyclic sieving phenomenon, *J. Combin. Theory Ser. A* **108** (2004), 17–50.

[2] B. Rhoades, Cyclic sieving, promotion, and representation theory, *J. Combin. Theory Ser. A* **117** (2010), 38–76.

[3] B. E. Sagan, The cyclic sieving phenomenon: A survey, *London Math. Soc. Lecture Note Ser.* **392**, Cambridge Univ. Press, Cambridge, 2011.

[4] J. Stembridge, Some hidden relations involving the ten symmetry classes of plane partitions, *J. Combin. Theory Ser. A* **68** (1994), 372–409.