Teacher Training University
Faculty of Mathematical Sciences

Time: 4 h

PHD QUALIFY EXAM, 1386

$\mathfrak{Commutative\ Algebra}$ 2

Throughout R is a commutative ring with $1 \neq 0$.

1. (a) Let $R \longrightarrow S$ be a faithfully flat ring homomorphism of noetherian rings, and let M be an R–module. Prove that:

$$\operatorname{Ass}_R(M) \subseteq \{\mathfrak{Q} \cap R : \mathfrak{Q} \in \operatorname{Ass}_S(M \otimes_R S)\}.$$

(b) Let $(R, \mathfrak{m})$ be a noetherian local ring and let M be a finitely generated R–module. Prove that if $\mathfrak{Q} \in \operatorname{Supp}_{\widehat{R}}(\widehat{M})$, then $\mathfrak{Q} \cap R \in \operatorname{Supp}_R(M)$ and that $\operatorname{ht}_M(\mathfrak{Q} \cap R) \leq \operatorname{ht}_{\widehat{M}}(\mathfrak{Q})$.

2. Assume that $(R, \mathfrak{m})$ is an artinian local ring. Prove that the following statements are equivalent.
(a) R is Gorenstien.
(b) $\mathfrak{a} = 0 :_R (0 :_R \mathfrak{a})$ for all ideals $\mathfrak{a}$ of R.
(c) $\mathfrak{a} \cap \mathfrak{b} \neq 0$ for all non-zero ideals $\mathfrak{a}$ and $\mathfrak{b}$ of R.

3. Assume that R is a regular local ring and that $\mathfrak{a}$ is an ideal of R of $\operatorname{ht} \mathfrak{a} = 1$. Prove that the following statements are equivalent.
(a) $R/\mathfrak{a}$ is Cohen–Macualay.
(b) $\operatorname{ht} \mathfrak{p} = 1$ for all $\mathfrak{p} \in \operatorname{Ass}(R/\mathfrak{a})$
(c) $\mathfrak{a}$ is a principal ideal.

4. Let $(R, \mathfrak{m})$ be a local ring and let $x_1, \cdots, x_n$ be a system of parameters for R. Prove that R is Cohen–Macaulay if and only if $x_1, \cdots, x_n$ is an R–sequence.

Now, assume that R is Cohen–Macualay and that $1 \leq i \leq n$. Prove that the ideal $(\sum_{j=1}^{i} Rx_j)^t$ is unmixed for all $t \in \mathbb{N}$.

5. Let $\mathfrak{q}$ be an $\mathfrak{m}$–primary ideal in the local ring $(R, \mathfrak{m})$ and suppose that $\mathfrak{q} = \cap_{i=1}^{n} \mathfrak{q}$ is a minimal decomposition of $\mathfrak{q}$ as intersection of irreducible ideals of R. Prove that n is an invariant (not dependent on the decomposition), which is called the type of $\mathfrak{q}$ and denoted by $r(\mathfrak{q})$.

Prove that if R is Cohen–Macualay, $\mathfrak{q}$ and $\mathfrak{q}'$ are generated by some sets of system of parameters then $r(\mathfrak{q}) = r(\mathfrak{q}')$.

6. Let $(R, \mathfrak{m})$ be a noetherian local regular ring of dimension d. Assume that $\mathfrak{a}$ is an ideal of R such that $\dim(R/\mathfrak{a}) = s$ and that $R/\mathfrak{a}$ is a regular ring.
(i) Prove that the dimension of the vector space $(\mathfrak{m}^2 + \mathfrak{a})/\mathfrak{m}^2$ over the field $R/\mathfrak{m}$ is $d - s$.
(ii) Prove that there exist $x_1, \cdots, x_{d-s}$ in $\mathfrak{a}$ such that $\{x_1, \cdots, x_{d-s}\}$ is a subset of a minimal generator of $\mathfrak{m}$.

(iii) Prove that $\mathfrak{a} = x_1R + \cdots + x_{d-s}R$.

Good Luck