Kharazmi University
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PhD comprehensive exam, Ordibehesht 1392

Homological Algebra II

Throughout, R is a commutative Noetherian ring with $1_R \neq 0$.

1. Let M be an R–module, $\mathfrak{a}$ an ideal of R. Prove that, for any $x \in R$ and each $i \in \mathbb{N}_0$, there exists an exact sequence
$$0 \longrightarrow \mathrm{H}^1_{Rx}(\mathrm{H}^{i-1}_{\mathfrak{a}}(M)) \longrightarrow \mathrm{H}^i_{\mathfrak{a}+Rx}(M) \longrightarrow \mathrm{H}^0_{Rx}(\mathrm{H}^i_{\mathfrak{a}}(M)) \longrightarrow 0.$$
(*Hint: You may first prove the exact sequence*
$$0 \longrightarrow H^0_{Rx}(M) \longrightarrow M \longrightarrow M_x \longrightarrow H^1_{Rx}(M) \longrightarrow 0.)$$

2. For an ideal $\mathfrak{a}$ of the ring R and an R–module M, define
$$\mathrm{cd}\,(\mathfrak{a}, M) := \sup\{i \in \mathbb{N}_0 | \mathrm{H}^i_{\mathfrak{a}}(M) \neq 0\}.$$
Prove that, for proper ideals $\mathfrak{a}$ and $\mathfrak{b}$ of R and a non–zero finitely generated R–module M,
$$\mathrm{cd}\,(\mathfrak{a}+\mathfrak{b}, M) \leq \mathrm{cd}\,(\mathfrak{a}, M) + \mathrm{ara}\,(\mathfrak{b}),$$
where $\mathrm{ara}\,(\mathfrak{b})$ denotes the arithmetic rank of $\mathfrak{b}$.

3. Let $(R, \mathfrak{m})$ be a local ring with $\dim R = d$. Prove that the following statements are equivalent.
   (i) $\mathrm{H}^d_I(R) = 0$ for all ideals I of R with $\dim\,(R/I) > 0$.
   (ii) $\mathrm{H}^d_{\mathfrak{p}}(R) = 0$ for all prime ideals $\mathfrak{p}$ of R with $\dim\,(R/\mathfrak{p}) = 1$.

4. Assume that $\mathfrak{a}$ is an ideal of R and that M is a finitely generated R–module. Prove the following statements.
   (a) If $R/\mathfrak{a}$ is Artinian, then $\mathrm{H}^i_{\mathfrak{m}}(M)$ is Artinian for all integers i.
   (b) If R is local, then the converse of (a) is also true.