*PhD comprehensive exam, Ordibehesht 1390*
*Tarbiat Moallem University, Faculty of Mathematical Sciences and Computer*
**Homological Algebra II**

Throughout all rings are commutative with $1 \neq 0$.

1. Let $R$ be a Noetherian ring and let $M$ be a finitely generated $R$–module. Assume that $\mathfrak{a}$ is proper ideal of $R$ which contains an $M$–regular sequence of length $n$. Prove the following statements.
   (a) If $R$ is local with maximal ideal $\mathfrak{m}$, then, for any positive integer $t$, $\operatorname{Hom}_R(R/\mathfrak{m}, \operatorname{Ext}^n_R(R/\mathfrak{a}, M)) \cong \operatorname{Hom}_R(R/\mathfrak{m}, \operatorname{Ext}^n_R(R/\mathfrak{a}^t, M))$.
   (b) $\operatorname{Ass}_R(\operatorname{H}^n_{\mathfrak{a}}(M)) \subseteq \operatorname{Ass}_R(\operatorname{Ext}^n_R(R/\mathfrak{a}, M))$.

2. Let $(R, \mathfrak{m})$ be a complete Noetherian local ring and let $I$ be an ideal of $R$. Set $t := \sup\{i : \operatorname{H}^i_I(R) \neq 0\}$. Assume that $\operatorname{H}^t_I(R)$ is Artinian.
   (a) Show that if $A$ is an Artinian $R$–module then
   $$\operatorname{Att}_R A = \{\mathfrak{p} \in \operatorname{Spec} R : \mathfrak{p} = \operatorname{Ann}_R(A/\mathfrak{p}A)\}.$$
   (b) Show that if $\mathfrak{p} \in \operatorname{Att}_R(\operatorname{H}^t_I(R))$ then $\dim R/\mathfrak{p} \geq t$.
   (c) Assume that $\mathfrak{p} \in \operatorname{Spec} R$ such that $\dim R/\mathfrak{p} = t$. Prove the following statements.
      (i) $\mathfrak{p} \in \operatorname{Att}_R(\operatorname{H}^t_I(R))$.
      (ii) $\operatorname{H}^t_I(R/\mathfrak{p}) \neq 0$.
      (iii) $\sqrt{I + \mathfrak{p}} = \mathfrak{m}$.

3. Let $R$ be a Noetherian ring and let $M$ be a finitely generated $R$–module. For an ideal $\mathfrak{a}$ of $R$, set $\operatorname{cd}(\mathfrak{a}, M) := \sup\{i \in \mathbb{N}_0 : \operatorname{H}^i_{\mathfrak{a}}(M) \neq 0\}$. Assume that $\operatorname{cd}(\mathfrak{a}, M) > 0$. Prove the following statements.
   (a) There is an integer $j$ such that $\operatorname{H}^j_{\mathfrak{a}}(M)$ is not finitely generated.
   (b) If we set $\operatorname{cd}(\mathfrak{a}) := \sup\{\operatorname{cd}(\mathfrak{a}, N) : N \text{is an } R\text{–module}\}$, then $\operatorname{cd}(\mathfrak{a}) = \operatorname{cd}(\mathfrak{a}, R)$.

4. Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let
   $$0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$$
   be an exact sequence of finitely generated $R$–modules. Assume that $d := \dim M$ and that $\operatorname{H}^d_{\mathfrak{m}}(K)$ is finitely generated. Show that if $\mathfrak{p} \in \operatorname{Supp} M$ with $\dim R/\mathfrak{p} = d$, then $\mathfrak{p} \in \operatorname{Ass} N$.

5. Suppose $(R, \mathfrak{m})$ is a Noetherian local ring. Show that for each finitely generated $R$–module $M$ and any $\mathfrak{p} \in \operatorname{Supp} M$,
   $$\operatorname{depth} M \leq \operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} + \dim R/\mathfrak{p} \leq \dim M.$$

Good luck.