PhD comprehensive exam, Ordibehesht 1390
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Commutative Algebra II

Throughout all rings are commutative with $1 \neq 0$.

1. Find an example for each of the following.
   (i) A regular ring which is not integral domain.
   (ii) A universally catenary ring which is not Cohen-Macaulay.
   (iii) A non–regular local ring $(R, \mathfrak{m})$ such that $R_{\mathfrak{p}}$ is regular for any $\mathfrak{p} \in \operatorname{Spec} R \setminus \{\mathfrak{m}\}$.
   (iv) A Gorenstein local ring which is not complete intersection.

2. Let R be a regular local ring and let I be a proper ideal of R. Prove the following statements.
   (a) If R/I is regular, then $I = \sum_{i=1}^{n} Rx_i$, for some $x_1, \cdots, x_n$ which is part of a minimal generating set for the maximal ideal of R.
   (b) R/I is Gorenstein if and only if $\operatorname{Ext}_R^i(R/I, R) \cong R/I$ whenever $i = \dim R - \dim R/I$, and $\operatorname{Ext}_R^i(R/I, R) = 0$ otherwise.

3. Assume that $(R, \mathfrak{m})$ is a Noetherian local ring and that M and N are finitely generated non–zero R–modules of finite projective dimensions. Assume that $\operatorname{Tor}_i^R(M, N) = 0$ for all $i > 0$. Prove that
   (i) $\operatorname{proj.dim}_R(M \otimes_R N) = \operatorname{proj.dim}_R M + \operatorname{proj.dim}_R N$,
   (ii) $\operatorname{depth}(M \otimes_R N) = \operatorname{depth} M - \operatorname{proj.dim}_R N$.

4. Let $(R, \mathfrak{m})$ be a regular local ring which is a subring of a Noetherian local ring $(A, \mathfrak{q})$ with $\mathfrak{m}A \subseteq \mathfrak{q}$. Prove that A is flat over R if and only if $\operatorname{grade}(\mathfrak{m}A, A) = \dim R$. Deduce that if A is Cohen-Macaulay and

$$\dim R + \dim A/\mathfrak{m}A = \dim A,$$

   then A is flat over R.

Good luck