Kharazmi University
Faculty of Mathematical Sciences &Computer
دانشگاه خوارزمی

PhD comprehensive exam, Ordibehesht 1392

Commutative Algebra II

Throughout all rings are commutative with $1 \neq 0$.

1. Let K be a field, $R = K[X_1, \cdots, X_n]$ the ring of polynomials in n indeterminates $X_1, \cdots, X_n$ with coefficients in K. Prove that, for any R–module M, $\operatorname{proj.dim}_R(M) \leq n$ and $\operatorname{inj.dim}_R(M) \leq n$.

2. Let $(R, \mathfrak{m})$ be a Noetherian local ring, M and N non–zero finitely generated R–modules. Let n be a positive integer.
   (a) Prove that $(0 :_R N)$ contains an M–regular sequence of length n if and only if $\operatorname{Ext}_R^i(N, M) = 0$, for all i, $i < n$.
   (b) Assume that there exist an M–regular sequence of length n in $(0 :_R N)$ and that $(0 :_R N)$ is an $\mathfrak{m}$–primary ideal. Prove that, for any prime ideal $\mathfrak{p}$ of R with $\dim(R/\mathfrak{p}) = 1$, one has $\operatorname{Ext}_R^i(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}, M_\mathfrak{p}) = 0$ for all i, $i < n - 1$.

3. Let $(R, \mathfrak{m})$ be a d dimensional Noetherian local ring.
   (a) Prove that if $\mathfrak{m} \notin \operatorname{Ass}_R(R/(x_1, \cdots, x_{d-1}))$ for all system of parameters $x_1, \cdots, x_d$ of R, then R is Cohen-Macaulay.
   (b) Assume that R is Cohen-Macaulay. Prove that, for any finitely generated R–module M, if $i \geq d$, then the ith syzygy module of a finite free resolution of M is a maximal Cohen-Macaulay R–module.

4. Let $(R, \mathfrak{m})$ be a Noetherian local ring, M an R–module. Prove that

$$\operatorname{depth} R \leq \operatorname{grade}_R(M) + \dim_R(M) \leq \dim R.$$