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Commutative Algebra 2

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Throughout all rings are commutative Noetherian with $1 \neq 0$.

I. Let A be a semilocal ring and let M be a finite A–module. Describe the three numerical invariants $\dim M$, $d(M)$ and $\delta(M)$ related to M. Prove that $\dim M = d(M) = \delta(M)$. (*Give your proof as complete as possible.*)

II. Let A be a ring, let M be a finite A–module and let I be an ideal of A such that $IM \neq M$. Prove that the following statements are equivalent:
(1) $\operatorname{Ext}^i_A(N, M) = 0$ for all $i < n$ and for any finite A–module N with $\operatorname{Supp}_A(N) \subseteq \mathrm{V}(I)$;
(2) $\operatorname{Ext}^i_A(A/I, M) = 0$ for all $i < n$;
(3) $\operatorname{Ext}^i_A(N, M) = 0$ for all $i < n$ and for some finite A–module N with $\operatorname{Supp}_A(N) = \mathrm{V}(I)$;
(4) there exists an M–sequence of length n contained in I.

III. Let $(A, \mathfrak{m})$ be a local ring and let M be a finite A–module. Prove that if $\text{inj.dim}\, M < \infty$ then $\text{inj.dim}\, M = \operatorname{depth} A$.

IV. Let A be a local ring and M be a non–zero finite A–module. Suppose that $\text{proj.dim}\, M < \infty$. Prove that

$$\text{proj.dim}\, M + \operatorname{depth} M = \operatorname{depth} A.$$

good luck, Mohammad T. Dibaei.