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PhD comprehensive exam, Ordibehesht 1392

Commutative Algebra III

Throughout, R is a commutative Noetherian ring with $1_R \neq 0$.

Definition. Let $(R, \mathfrak{m})$ be a local ring. A finitely generated R–module M is called *Gorenstein* if it is maximal Cohen-Macaulay and has finite injective dimension.

1. Let $(R, \mathfrak{m})$ be a homomorphic image of a Gorenstein local ring. Prove the following statements.
   (a) If $\operatorname{inj.dim}(M) < \infty$, then M is a homomorphic image of a Gorenstein R–module.
   (b) If $\operatorname{inj.dim}(M) < \infty$, then there exists an exact sequence
   $$0 \longrightarrow M_s \longrightarrow M_{s-1} \longrightarrow \cdots \longrightarrow M_0 \longrightarrow M \longrightarrow 0,$$
   where M_i is a Gorenstein R–module for all i, $0 \leq i \leq s$.

2. Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring. Prove that if M is a Gorenstein R–module and N is a finitely generated R–module of finite projective dimension, then
   $$\operatorname{Tor}_i^R(M, N) = 0$$
   for all $i > 0$.

3. Let $R = \bigoplus_{n \in \mathbb{Z}} R_n$ be a graded ring such that $(R_0, \mathfrak{m}_0)$ is local with infinite residue field $R_0/\mathfrak{m}_0$. Assume that I is a graded ideal of R generated by elements of degree 1. Prove that if $\mathfrak{p}_1, \cdots, \mathfrak{p}_n$ are prime ideals of R such that $I \not\subseteq \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n$, then $I \cap R_1 \not\subseteq \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n$.

4. Let $R = \bigoplus_{n \geq 0} R_n$ be a graded ring and let M be a finitely generated R–module. Prove that there is a homogeneous element $x \in R_+$ such that $(0 :_M x)_n \neq 0$ only for finitely many integers n (such element is called *superficial* for M). Show that if $R = R_0[R_1]$ and the residue field of R_0 is infinite, then we may choose $x \in R_1$.

   Prove that if $\dim_R M > \dim R_0$, then $\dim_R(M/xM) = \dim_R M - 1$ for all superficial element $x \in R_+$ for M.