Teacher Training University
Faculty of Mathematical Sciences

Time: 4 h

PHD QUALIFY EXAM, 1386

𝔥omological 𝔄lgebra 2

Throughout R is a commutative ring with $1 \neq 0$.

1. Let $(R, \mathfrak{m})$ be a noetherian local ring and let M be an R–module . Denote $\mu^r(M) := \dim_{R/\mathfrak{m}} \operatorname{Ext}_R^r(R/\mathfrak{m}, M)$.

i) Prove that, for any ideal $\mathfrak{a}$ of R and any R–module M,

$$\mu^2(M) \leq \mu^0(\mathrm{H}^2_{\mathfrak{a}}(M)) + \mu^1(\mathrm{H}^1_{\mathfrak{a}}(M)) + \mu^2(\mathrm{H}^0_{\mathfrak{a}}(M)).$$

ii) Assume that M is a finitely generated R–module and that $\mathfrak{a}$ is an ideal generated by an M–regular sequence of length 2. Prove that $\mu^0(\mathrm{H}^2_{\mathfrak{a}}(M)) = \mu^2(M)$ and $\mu^1(\mathrm{H}^2_{\mathfrak{a}}(M)) = \mu^3(M)$. What is your idea about $\mu^r(\mathrm{H}^2_{\mathfrak{a}}(M)) = \mu^{r+2}(M)$ for all $r \geq 0$. Write down any comment, proof, or disproof about it.

2. Assume that R is a noetherian ring and that M is a finite R–module. Denote $\mu^i(\mathfrak{p}, M) := \dim_{k(\mathfrak{p})} \operatorname{Ext}^i_{R_{\mathfrak{p}}}(k(\mathfrak{p}), M_{\mathfrak{p}})$. Prove that if $\mathfrak{p} \subset \mathfrak{q}$ are distinct prime ideals of R with no other prime ideals between them, then

$$\mu^i(\mathfrak{p}, M) \neq 0 \Longrightarrow \mu^{i+1}(\mathfrak{p}, M) \neq 0.$$

3. Let $\mathfrak{a}$ be an ideal of a noetherian ring R and let M be an R–module. Assume that s is a non–negative integer and that $\operatorname{Ext}^j_R(R/\mathfrak{a}, \mathrm{H}^i_{\mathfrak{a}}(M))$ is finitely generated for all $j \geq 0$ and all i, $0 \leq i < s$. Prove that $\operatorname{Ext}^s_R(R/\mathfrak{a}, M)$ is finitely generates if and only if $\operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}^s_{\mathfrak{a}}(M))$ is finitely generated.
(Here is a hint: set $E := E(M/\Gamma_{\mathfrak{a}}(M))$ and apply an induction argument on s.)

4. Let R be a noetherian ring.

(i) Assume that A is a representable R–module. Show that $M \otimes_R A$ is representable.

(ii) Let $f : R \longrightarrow T$ be a ring homomorphism and let A be a representable T–module. Show that A is a representable R–module and

$$\operatorname{Att}_R(A) = \{f^{-1}(\mathfrak{p}) : \mathfrak{p} \in \operatorname{Att}_T(A)\}.$$

(ii) Let M be an R–module with $\dim_R(M) < \infty$ and $\dim(R/\operatorname{Ann}_R(M)) = \dim_R(M)$. Show that, for any ideal $\mathfrak{a}$ of R,

$$\mathrm{H}^n_{\mathfrak{a}}(M) \cong \mathrm{H}^n_{\frac{\mathfrak{a}+\operatorname{Ann}_R(M)}{\operatorname{Ann}_R(M)}}(R/\operatorname{Ann}_R(M)) \otimes_R M,$$

where $n = \dim_R(M)$. Deduce that $\mathrm{H}^n_{\mathfrak{a}}(M)$ is a representable R–module.

5. Let $(R, \mathfrak{m})$ be a noetherian local ring of dimension n and suppose that $\mu(\mathfrak{m}) := 1 + \operatorname{depth}(R)$. Prove that $\mathrm{H}^n_{\mathfrak{m}}(R) = E(R/\mathfrak{m})$. (Hint: Induction on $\operatorname{depth}(R)$.)

6. Let k be a field and let $R = k[[x, y, u, v]]/(xu - yv)$, and set $I := (x, y)R$. Prove that $\mathrm{H}^3_I(R) = 0$ but $\mathrm{H}^3_I(R) \neq 0$.

GOOD LUCK