PhD comprehensive exam, Ordibehesht 1390
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Homological Algebra III

Throughout all rings are commutative with $1 \neq 0$.

1. Assume that $(R, \mathfrak{m})$ is a complete Noetherian local ring and M is a finitely generated R–module. Let $i \geq 0$ be an integer and let $\mathfrak{p} \in \operatorname{Att} \mathrm{H}^i_{\mathfrak{m}}(M)$. Prove that following statements.
   (i) $\dim R/\mathfrak{p} = \dim M$ if and only if $i = \dim M$.
   (ii) If $\operatorname{Min} M = \{\mathfrak{p} \in \operatorname{Supp} M : \dim R/\mathfrak{p} = \dim M\}$, then $\operatorname{ht} \mathrm{a}(M) > 0$, where $\mathrm{a}(M) = \bigcap_{i<\dim M} (0 :_R \mathrm{H}^i_{\mathfrak{m}}(M))$.

2. Let $(S, \mathfrak{n}, k)$ be a complete Cohen-Macaulay local ring and let $(R, \mathfrak{m}, k)$ be a homomorphic image of S. Assume that M is a finitely generated R–module. Show that
$$\operatorname{Hom}_R(\mathrm{H}^i_{\mathfrak{m}}(M), \mathrm{E}_R(k)) \cong \operatorname{Hom}_S(\mathrm{H}^i_{\mathfrak{n}}(M), \mathrm{E}_S(k)).$$
   Also prove that
$$\operatorname{Hom}_R(\mathrm{H}^i_{\mathfrak{m}}(M), \mathrm{E}_R(k)) \cong \operatorname{Ext}^{d-i}_S(M, \omega_S),$$
   where $d = \dim S$ and ω_S is the canonical module of S.

3. Assume that $(R, \mathfrak{m})$ is a local ring. Let $\mathfrak{a}$ be a proper ideal of R and let M be a finitely generated R–module.
   (i) Prove that if $\mathfrak{p} \in \operatorname{Ass} M$ is such that $\operatorname{ht}(\frac{\mathfrak{a}+\mathfrak{p}}{\mathfrak{p}}) = 1$, then $f_{\mathfrak{a}}(M) = 1$.
   (ii) Suppose that $\mathfrak{b}$ is a second proper ideal of R and that
$$\min(\dim R/\mathfrak{a}, \dim R/\mathfrak{b}) > \dim \frac{R}{\mathfrak{a} + \mathfrak{b}}.$$
   Show that if R is complete local domain, then
$$\operatorname{ara}(\mathfrak{a} \cap \mathfrak{b}) \geq \dim R - \dim \frac{R}{\mathfrak{a} + \mathfrak{b}} - 1.$$

4. Assume that R is a Noetherian ring and that $\mathfrak{a}$ is a proper ideal of R.
   (i) Show that $\mathrm{H}^n_{\mathfrak{a}}(R) \neq 0$ for some integer $n \geq 0$.
   (ii) Assume that $\mathfrak{p} \in \operatorname{Supp}(\mathrm{H}^{\operatorname{ht}\mathfrak{p}}_{\mathfrak{a}}(R))$. Prove that for any $\mathfrak{Q} \in \operatorname{Spec} \widehat{R_{\mathfrak{p}}}'$ such that $\dim(\widehat{R_{\mathfrak{p}}}'/\mathfrak{Q}) = \operatorname{ht}\mathfrak{p}$, we have $\dim \frac{\widehat{R_{\mathfrak{p}}}'}{\mathfrak{a}\widehat{R_{\mathfrak{p}}}' + \mathfrak{Q}} = 0$, where
$$\widehat{R_{\mathfrak{p}}}' := \varprojlim_n R_{\mathfrak{p}}/\mathfrak{p}^n R_{\mathfrak{p}}.$$

5. Let $(R, \mathfrak{m})$ be a Noetherian local ring and let M be a finitely generated R–module. Prove that if $\mathfrak{p} \in \operatorname{Ass} M$ then $\mathfrak{p} \in \operatorname{Att}(\mathrm{H}^{\dim R/\mathfrak{p}}_{\mathfrak{m}}(M))$. (*Give the proof in detail!*)

GOOD LUCK