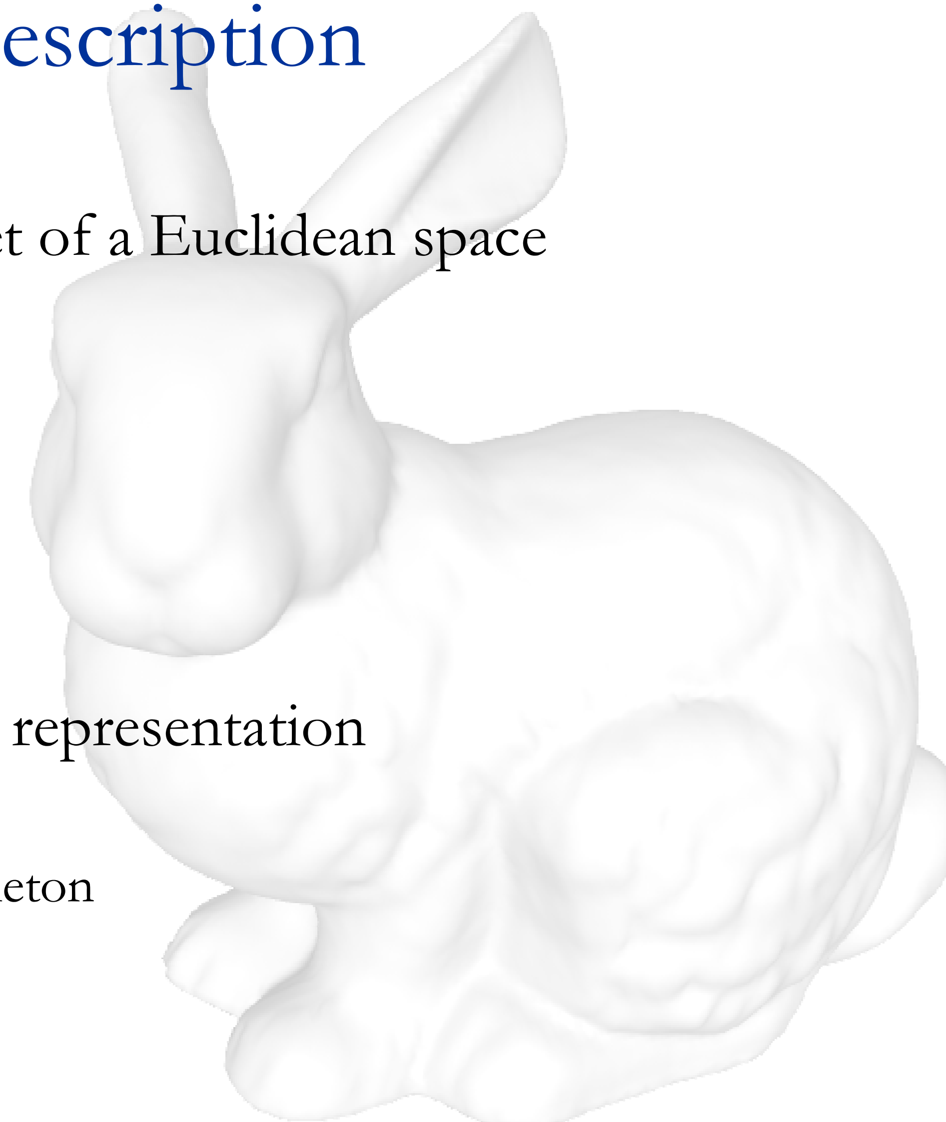
Shape Description via Persistent Homology

Afra Zomorodian

Department of Computer Science

Stanford University

Shape Description



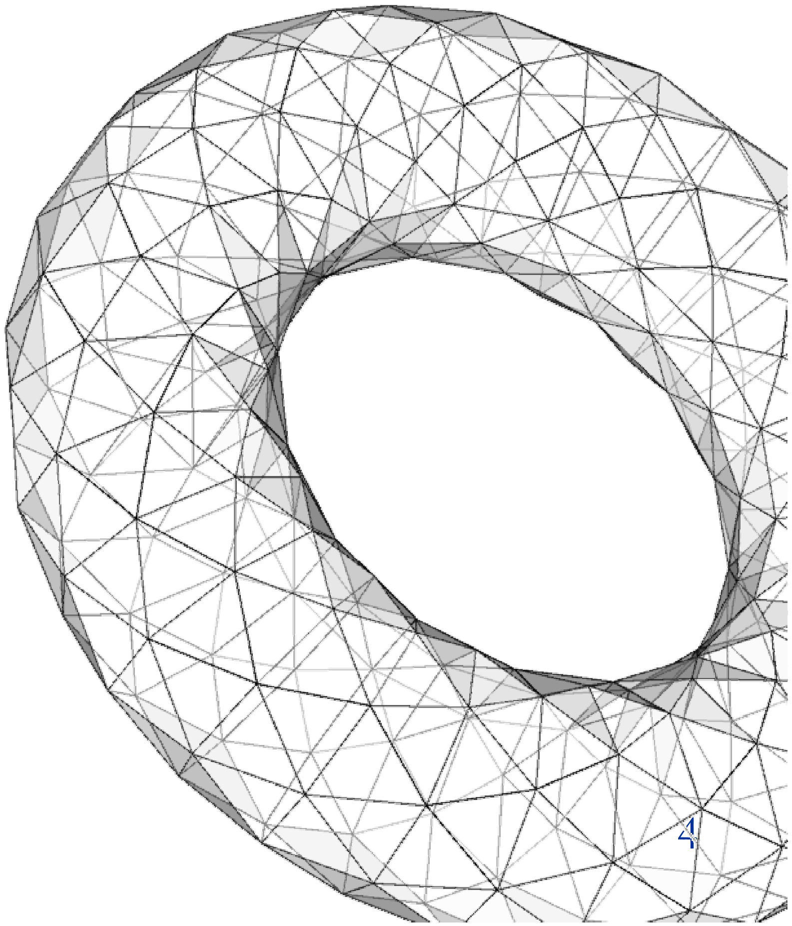
- A shape is any closed subset of a Euclidean space
- Questions:
  - Feature identification
  - Matching
  - Classification
  - Parameterization of a family
- Shape descriptor: compact representation
  - Moment invariants
  - Medial axis transform or skeleton
  - Fourier parametrization
  - Shape distributions
  - Multiresolution Reeb graphs

Invariants

- An invariant is a map that assigns the same object to shapes of the same *type*.
- Shape invariants (Felix Klein, Erlanger Programm, 1872)
  1. *Transform* space in a fixed way
  2. *Observe* properties that do not change
- Rigid motions: Euclidean geometry
- Smooth diffeomorphisms: topology

Geometry versus Topology

- Euclidean geometry
  - What does the shape look like?
  - Local
  - Quantitative
  - Low-level
- Topology
  - How is a shape connected?
  - Global
  - Qualitative
  - High-level



Homology

- An algebraic invariant that captures how shapes are connected
- For a space X, there is a group $H_k(X)$ in each dimension k
- The rank β_k of $H_k(X)$ is the kth Betti number
- The Betti numbers count topological attributes:
  - β_0: # components
  - β_1: # tunnels or loops
  - β_2: # voids

F	A	B
$\beta_1 = 0$	$\beta_1 = 1$	$\beta_1 = 2$

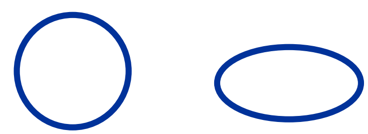
Coarse Invariant

- Cannot tell difference between a 'U' and a 'V'
- Sharp features

U V

$\beta_1 = 0$ $\beta_1 = 0$

- Cannot tell difference between a circle and an ellipse
- Soft features



$\beta_1 = 1$ $\beta_1 = 1$

Approach

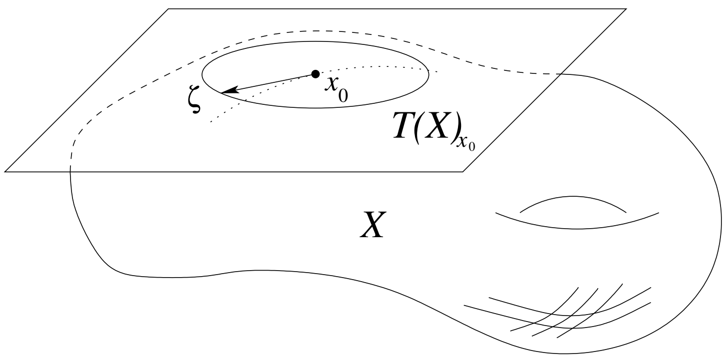
- Geometry: *differentiating* power
- Topology: *classifying* power
- Idea: combine the two
  - Apply topology to a *derived* space
  - Enrich derived space with geometric information
- Tools
  - Tangent Complex
  - Filtered by curvature
  - Persistent Homology
- Deliverables
  - Barcode shape descriptor
  - Metric over the space of barcodes

Overview

- Theory
  - Tangent Complex
  - Persistence
  - Filtered Tangent Complex
  - Mathematical Objects
- Practice
  - Point Cloud Data
  - Approximating the Tangent Complex
  - Shape Classification
  - Parameterization of Shapes

Tangent Complex

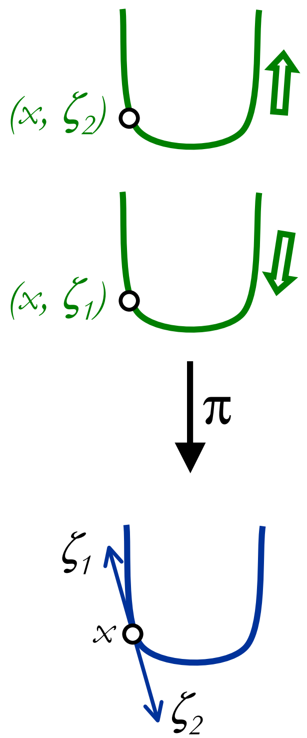
- Let $X \subseteq \mathbb{R}^n$ be a space.
- We pair a tangent ζ at a point $x \in X$ to get $T^0(X) \subseteq X \times \mathbb{S}^{n-1}$



$$T^0(X) = \left\{ (x, \zeta) \mid \lim_{t \to 0} \frac{\mathrm{d}(x + t\zeta, X)}{t} = 0 \right\}$$

- The tangent complex $T(X)$ of X is the closure of T^0
- The projection $\pi : T(X) \to X$ projects (x, ζ) to its basepoint x
- Basepoint $x \in X$ has fiber $T(X)_x = \pi^{-1}(x) \in \mathbb{S}^{n-1}$

Curve Example

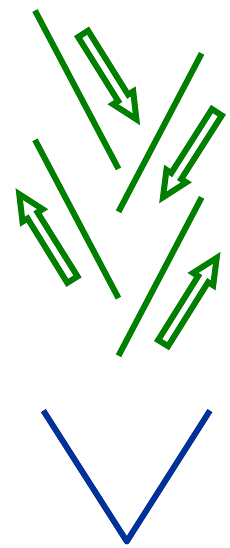


T(*X*) has two components: $\beta_0(T(X)) = 2$

There are two points in its fiber $\pi^{-1}(x)$ in the tangent complex *T*(*X*)

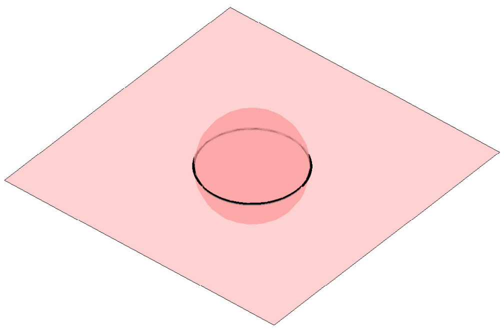
Every point *x* on a smooth curve *X* has two tangent directions.

A corner point has four tangent directions: $\beta_0(T(X)) = 4$

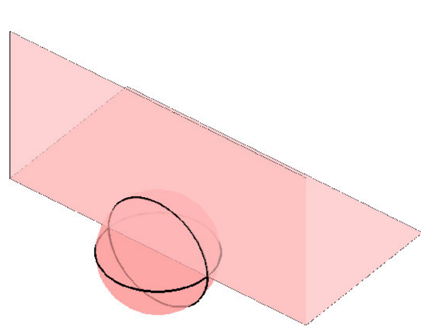


Surface Fibers

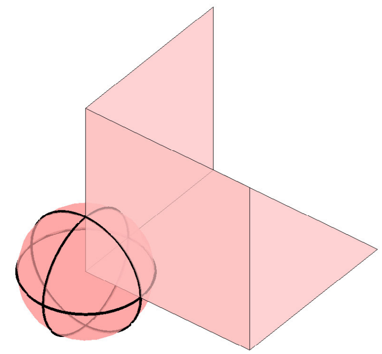
- The fiber for a point on a surface is a set of circles that intersect pairwise



Smooth: 1 circle
or $\mathbb{S}^1$

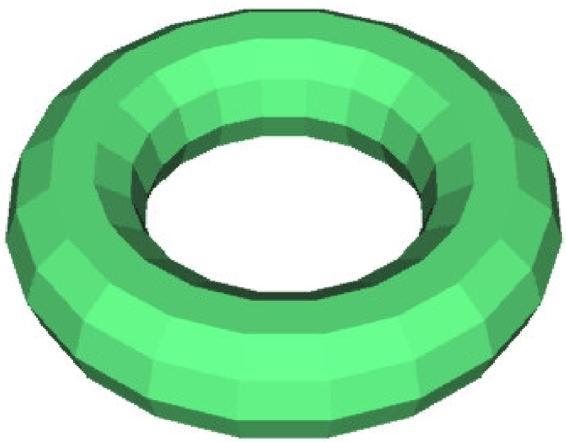


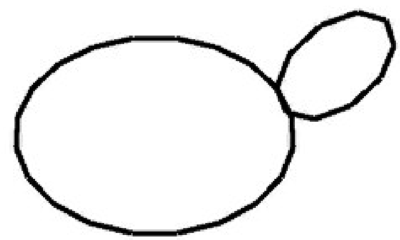
Edge: 2 circles
or $(\mathbb{S}^1)^3$



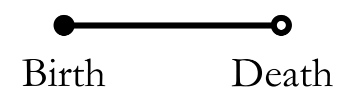
Cone: 3 circle
or $(\mathbb{S}^1)^7$

Persistence





- As the space evolves, topological attributes are created and destroyed, each having a lifetime of existence.

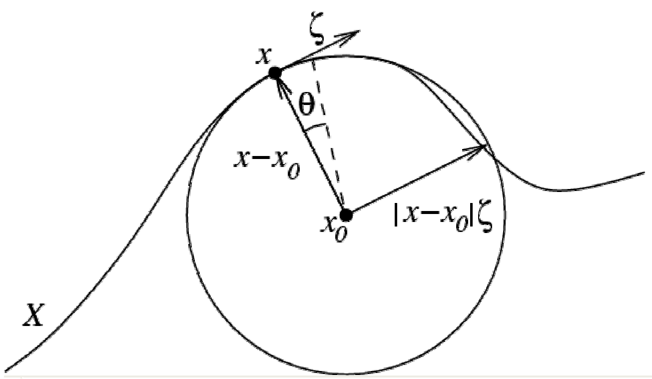


- Persistent Homology gives us these lifetimes as a collection of intervals, a persistence barcode.

Second Order Contact

- We incrementally construct the tangent complex by looking at the curvature at each point.

- $(x, \zeta) \in T^0(X)$
  has a circle of second order contact
  if $\exists\, x_0 \in \mathbb{R}^n$ such that
  $(x - x_0) \cdot \zeta = 0$ and



$$\lim_{\theta \to 0} \frac{\mathrm{d}(x_0 + \cos\theta \cdot (x - x_0) + \sin\theta \cdot |x - x_0| \cdot \zeta, X)}{\theta^2} = 0.$$

Filtered Tangent Complex

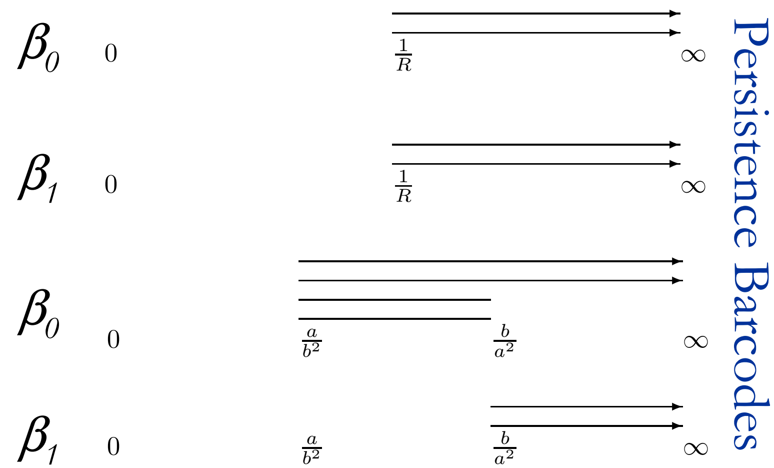
- Define $\rho(x, \zeta) = |x - x_0|$ to be the radius the circle of second order contact, if it exists
- Let $T_\delta{}^0(X)$ be points $(x, \zeta) \in T^0(X)$ that have circle of second order contact with $1/\rho(x, \zeta) \leq \delta$.
- Let $T_\delta(X)$ be the closure of $T_\delta{}^0(X)$.

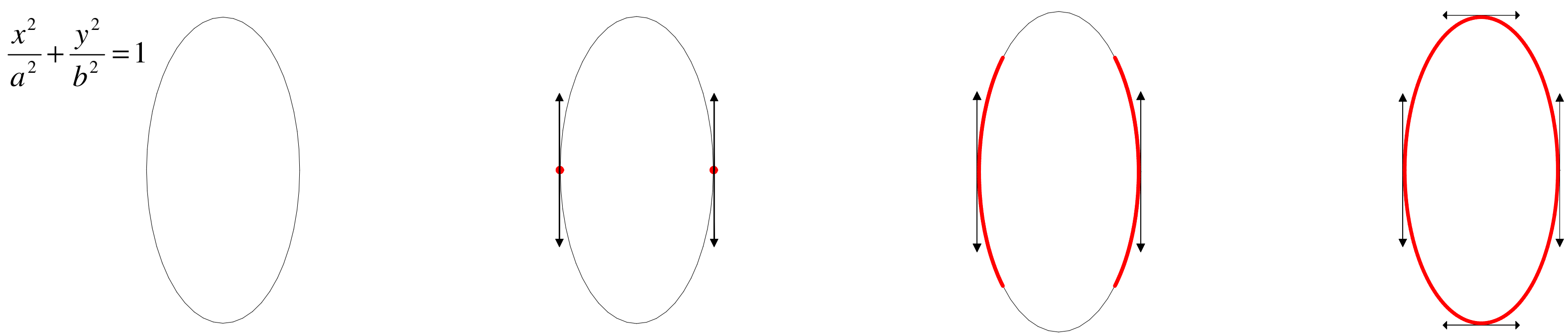
- The filtered tangent complex $T^{filt}(X)$ is the family $\{T_\delta(X)\}_{\delta \geq 0}$.
- The tame tangent complex $T^{tame}(X)$ is $\bigcup_\delta T_\delta(X)$.

Differentiating a Circle from an Ellipse

T^{filt}(circle of radius R) is simple: the entire complex (2 copies of circle) appears when $\delta = 1/R$.

T^{filt}(ellipse) evolves through a stage with four components: points at lower curvature appear earlier.



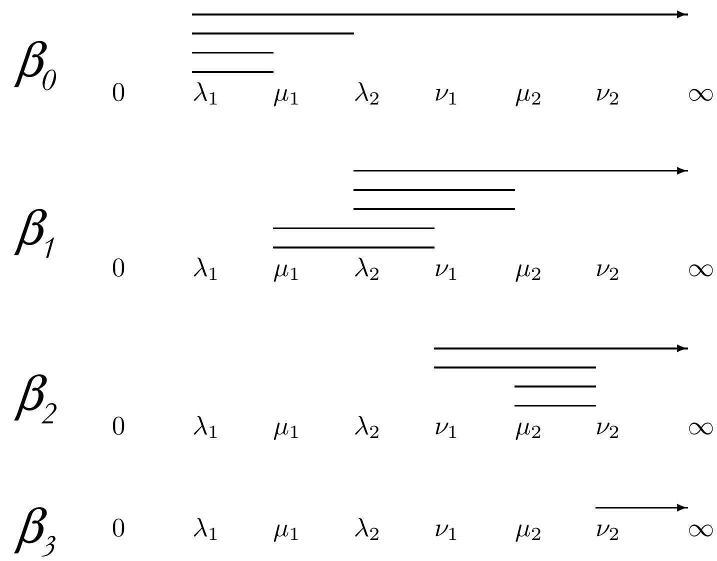


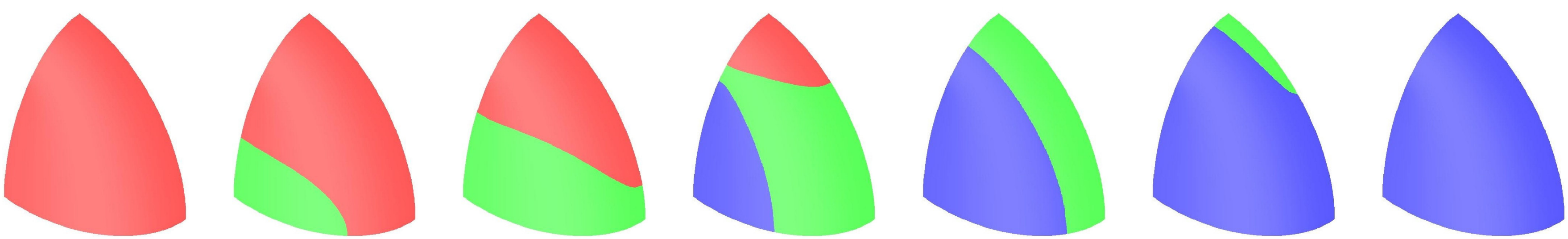
$0 \leq \delta < \frac{a}{b^2}$ $\quad$ $\delta = \frac{a}{b^2}$ $\quad$ $\frac{a}{b^2} < \delta < \frac{b}{a^2}$ $\quad$ $\delta \geq \frac{b}{a^2}$

Describing an Ellipsoid

- The evolution of T^{filt} depends upon the ordering of six critical values: the principal curvatures κ_1 and κ_2 at the ellipsoid extrema

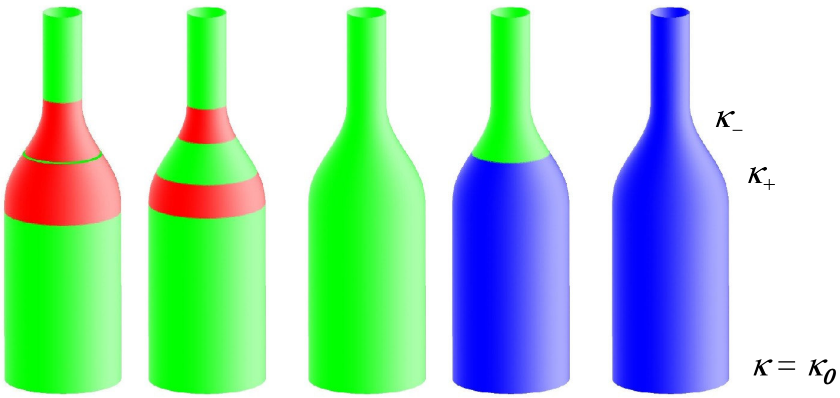
- We show the case $a^2c < b^3 < ac^2$ for ellipsoid $$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$





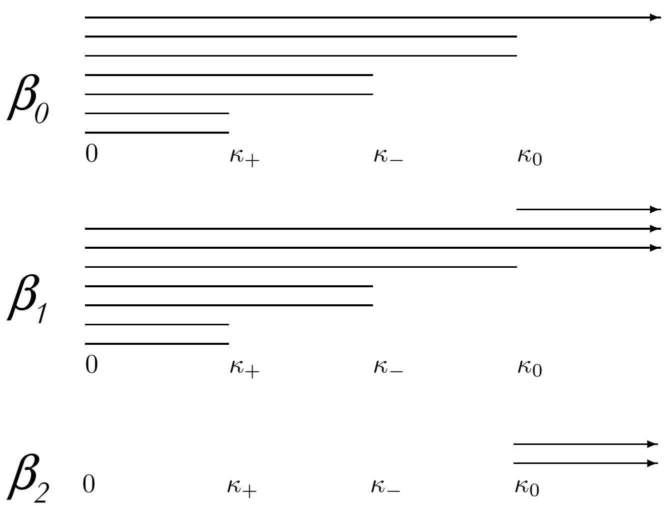
(empty, two arcs, circle)

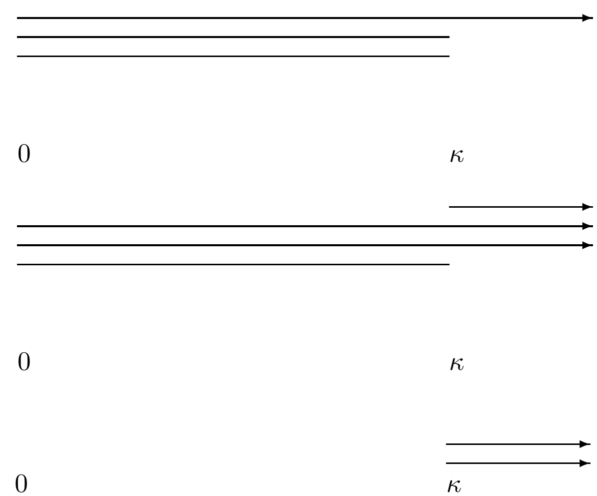
Differentiating a Bottle from a Glass



(empty, two arcs, circle)

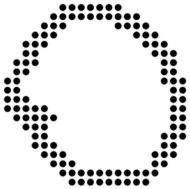
- We model the glass as the bottom of the bottle.
- The evolution of T^{filt} depends on neck curvatures κ_+, κ_- and cross-sectional curvature $\kappa = \kappa_0$ at the bottom.

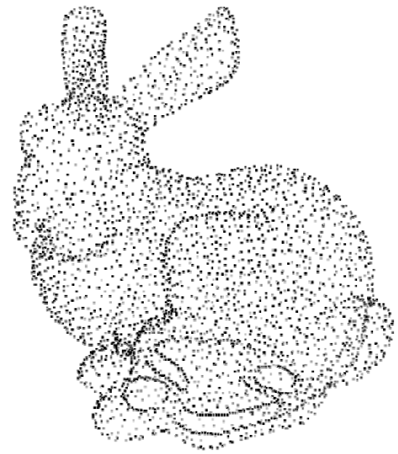


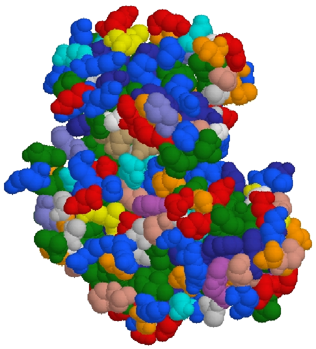


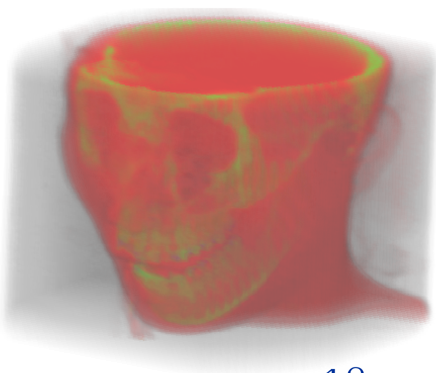
Point Cloud Data

- Nearly all shape data
  - 2D: images
  - 3D: surface data, molecules
  - *n*D: shape spaces
- Traditional approach: construct useful structure
  - Vision: OCR
  - Graphics: surface reconstruction
  - CAD: conversion to B-spline surfaces
- Recent approach: first class object
  - Shape representation
  - Rendering
  - Modeling

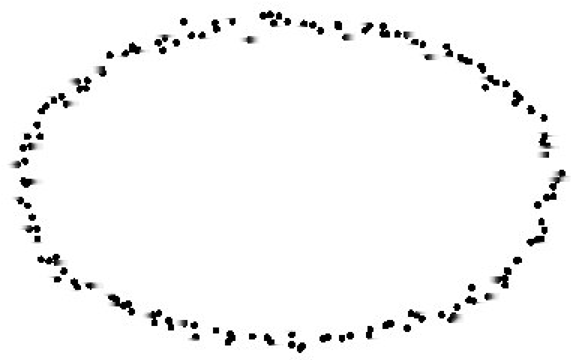






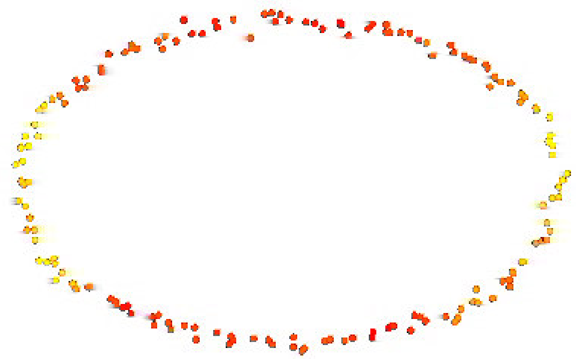


Fibers



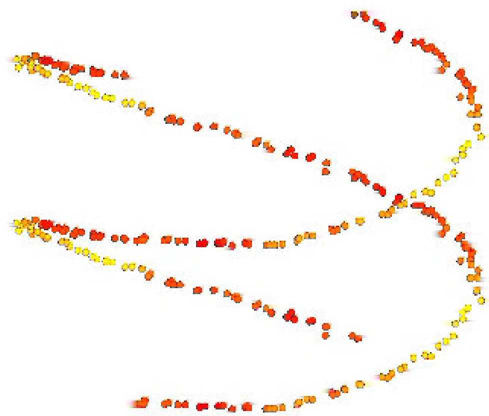
- PCD $P \subset X$
- We compute fibers $\pi^{-1}(P)$ by Total Least Squares using k-nearest neighbors

Filtering by Curvature



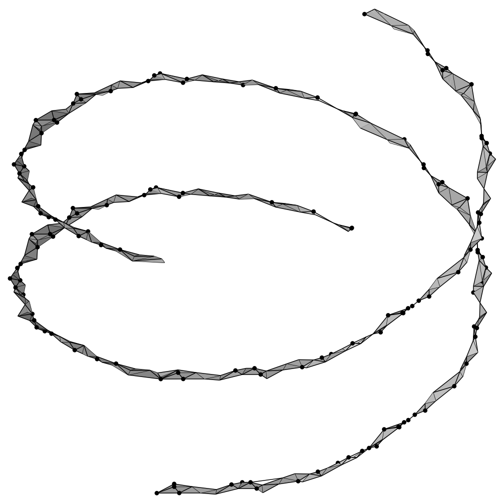
- Transform points to coordinate frame provided by tangent computation
- Fit osculating parabola instead of osculating circle

Approximating *T(X)*



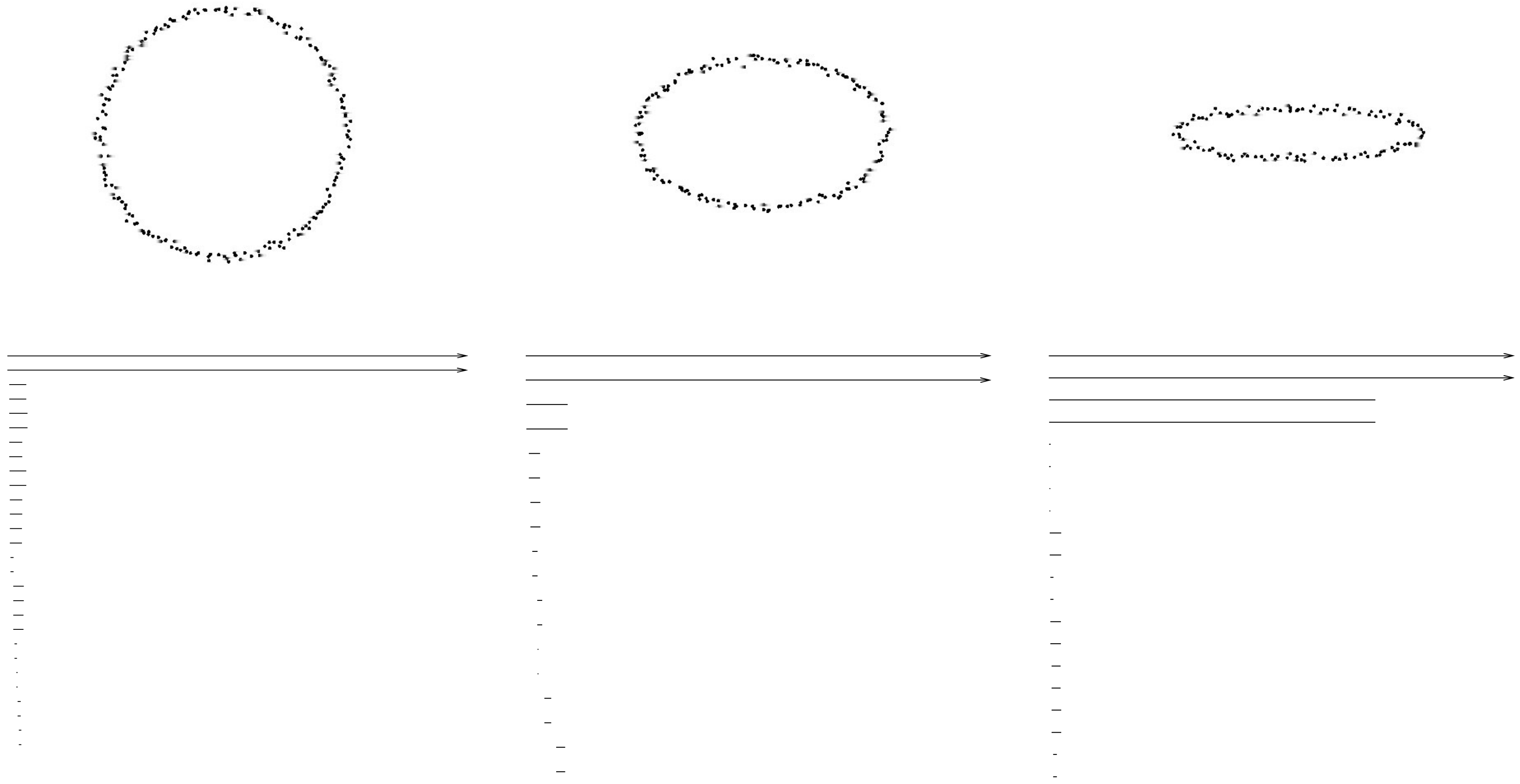
- $\mathbb{R}^n \times \mathbb{S}^{n-1}$ as $ds^2 = dx^2 + \omega^2\, d\zeta^2$
- $d^2(\tau,\tau') = \sum_{i=1}^{n}(x_i - x_i')^2 + \omega^2 \sum_{i=1}^{n}(\zeta_i - \zeta_i')^2$
- $T(X) \approx \bigcup_{p \in \pi^{-1}(P)} B_\varepsilon(p)$

Complex



- Rips complex
- $R_\varepsilon(M) = \{\text{conv } T \mid T \subseteq M, \text{d}(s, t) \leq \varepsilon, s,t \in T\}$

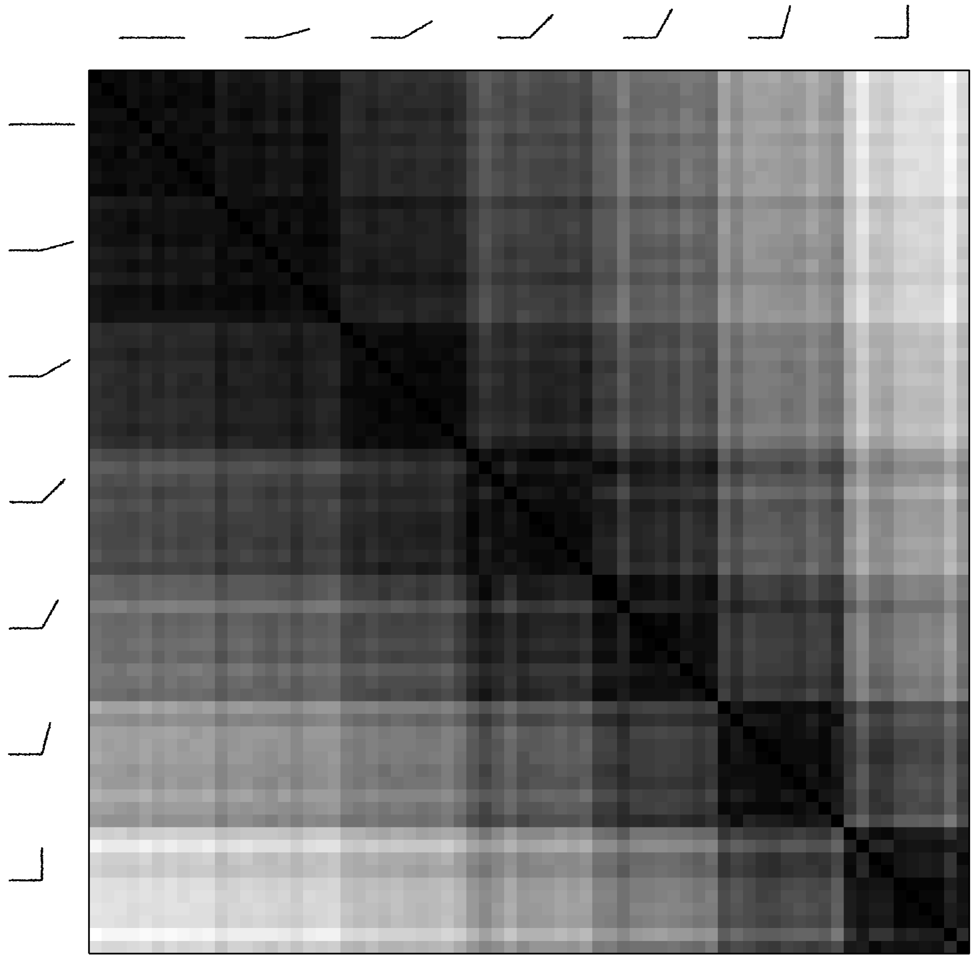
Family of Ellipses



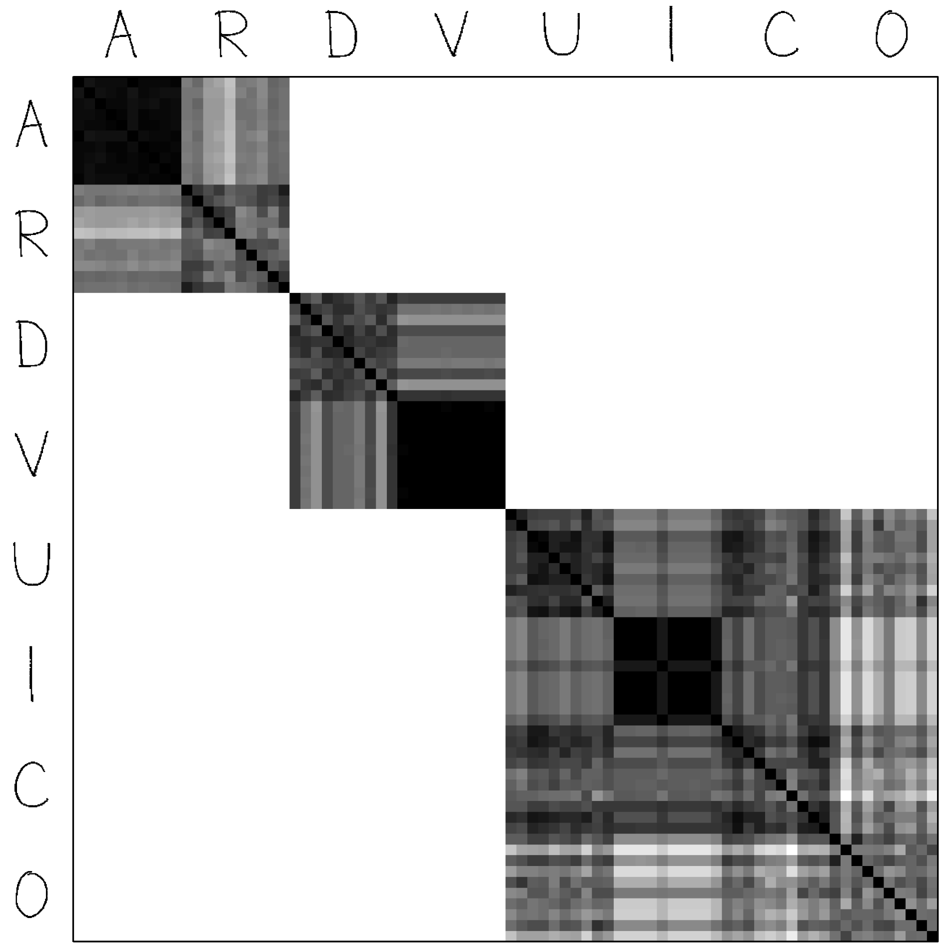
Barcode Metric

- Let I, J be any two intervals in a barcode.
- We define their dissimilarity $\delta(I, J) = |I \cup J - I \cap J|$
- Given a pair of barcodes B_1 and B_2
- A matching is a set $M(B_1, B_2) \subseteq B_1 \times B_2$, so that any interval in B_1 or B_2 occurs in at most one pair (I, J)
- Let M_1, M_2 be the matched intervals from B_1, B_2 and N be the unmatched intervals in the matching
- Let $d_M(B_1, B_2) = \sum_{(I, J) \in M} \delta(I, J) + \sum_{L \in N}\{|L|\}$
- Metric: $D(B_1, B_2) = \min_M d_M(B_1, B_2)$

Articulated Arm



Letter Classification



Conclusion

- Apply persistent homology to geometry-rich derived space to get a compact shape descriptor called barcode
- Provide a metric over the space of all barcodes
  - Comparison
  - Matching
  - Classification
- Compute tangent complexes for Curve Point Cloud Data
- Future Work:
  - Surface PCD
  - Complexes with lower intrinsic dimension