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On the Controllability of a Quantized Maxwell-Lorentz Equation

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One can easily see that a single classical oscillator with a control term u:

$$\ddot{x} = -\omega^2(x-u), \; x \in \mathbb{R}, \quad u \in \mathbb{R}$$

is controllable. In a recent paper (M. Mirrahimi and P. Rouchon, IEEE-AC, May 2004) it is proved that, contrary to such finite dimensional classical system the PDE corresponding to a quantum harmonic oscillator:

$$\iota \frac{d}{dt}\Psi = \frac{1}{2m}\frac{\partial^2}{\partial x^2} + \frac{m\omega^2}{2}(x^2 - 2ux)$$

is not controllable: its controllable part is of dimension 2 and corresponds to the dynamic of the average position. More generally, for the quantum harmonic oscillator of any dimension, similar lacks of controllability occur whatever the number of control is: the controllable part still corresponds to the average position.

We propose here an extension to a more complicated physical model corresponding to a Maxwell-Lorentz equation where the control term appears in the currents source j. The classical problem is to control a wave equation of the form

$$\frac{1}{c^2}\ddot{E} = \triangle E - \frac{1}{\epsilon_0 c^2}\frac{\partial}{\partial t}j \quad \text{in} \quad (0,T)\times\Omega$$

$$E(t,.) = 0 \quad \text{in} \;\; \partial\Omega$$

$$E(t=0,.) = E_0(.) \;\; , \;\; \frac{\partial}{\partial t}E(t=0,.) = E_1(.) \quad \text{in} \quad \Omega$$

where Ω is a bounded domain in $\mathbb{R}^3$ and j corresponding to localized currents plays the role of the control u. The controllability of such a system has been widely studied (see, for example, a recent paper of E. Zuazua and X. Zhang in the Proceedings of the Second

IFAC Workshop on Lagrangian and Hamiltonian methods in Nonlinear Control, Sevilla, 2003 and the references herein).

We consider here the quantified dynamics of the electromagnetic field E, the current source j, i.e., the control, remaining classical. In the quantizing process the system becomes nonlinear. We study its controllability using the Heisenberg picture (unitary transformations). This method is widely used by physicists and simplifies the computations, with respect to the PDE language (see, e.g., C. Cohen-Tannoudji et al., Photons and Atoms: Introduction to Quantum Electrodynamics). We prove that the controllable part is included in the classical Maxwell-Lorentz equations that describe the evolution of the mean value of the operator associated to electric field.