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Free Boundary Problems of Obstacle Type

(3 Lectures)

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Lecture 1

In these lectures we discuss the free boundary for the obstacle-type problem

$$(\Delta u - f)u = 0$$

(with $f \in C^1$).

At the first lecture we start with the most elementary cases, give some examples and motivation.

Those problem that we will be concerned with are *Classical Obstacle problem*, and somewhat general versions such as the problem of *harmonic continuation of potentials*, *Problems in super-conductivity, two membrane problem* and in general *minimization*

$$\int_\Omega (|\nabla u|^2 + \lambda_+ \max(u,0) + \lambda_- \max(-u,0)dx.$$

The latter give rise to the following type of solutions.

$$\Delta u = \frac{\lambda_+}{2}\chi_{\{u>0\}} - \frac{\lambda_-}{2}\chi_{\{u<0\}},$$

where $\lambda_+ > 0$ and $\lambda_- > 0$,

In this regard you can download articles at:

http://www.math.kth.se/~ henriksh/general/publication.html

Lecture 2

At this second lecture we discuss regularity for the solutions to the obstacle problem as well as those related.

Standard regularity for solutions are obtained mostly from classical methods. However, to obtain the optimal regularity one needs to use more elaborated technique.

The best regularity for solutions to

$$\Delta u = g(x, u)$$

with g bonded is of course $C^{1,\alpha}, \alpha < 1$.

Observe that the obstacle problem can be written in this way with $g(x,u) = f(x)H(u)$, where H is now the Heaviside function.

In this second lecture we prove optimal $C^{1,1}$ regularity for solutions to all problems mentioned in the first lecture.

You can find some notes in next pages.

In this regard you can download articles at:

http://www.math.kth.se/~ henriksh/general/publication.html, and

http://www.ipm.ac.ir/pde2004/shahgholian.pdf

Lecture 3

In this lecture we discuss regularity issues for the free boundaries that arises in the problems mentioned earlier.

We first show the regularity of the free boundary for the obstacle problem. This is done classically by L. Caffarelli, and can be found at:

http://www.ma.utexas.edu/users/combs/Caffarelli/obstacle.pdf

Next, if time allows, we show some new techniques to prove regularity for the other problems. You may see the following article:

http://www.arxiv.org/PS-cache/math/pdf/0010/0010016.pdf or

http://www.emis.de/journals/Annals/151-1/caffarel.pdf

In this regard you can download articles at:

http://www.math.kth.se/~ henriksh/general/publication.html