Spectral Theory and Geometry

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1 Lecture 1 : Introduction and basic examples

1.1 The case of functions

Let (M, g) be a smooth, connected and C^∞ Riemannian manifold with boundary ∂M. The boundary is a Riemannian manifold with induced metric $g_{|\partial M}$. We suppose ∂M to

be smooth. We refer to the book of Sakai [Sa] for a general introduction to Riemannian Geometry and to Bérard [Be] for an introduction to the spectral theory.

For a function $f \in C^2(M)$, we define the Laplace operator or Laplacian by

$$\Delta f = \delta df = -div\ gradf$$

where d is the exterior derivative and δ the adjoint of d with respect to the usual L^2-inner product

$$(f, h) = \int_M fh\ dV.$$

In local coordinates $\{x^i\}$, the laplacian reads

$$\Delta f = -\frac{1}{\sqrt{det(g)}} \sum_{i,j} \frac{\partial}{\partial x_j}(g^{ij}\sqrt{det(g)}\frac{\partial}{\partial x_i} f).$$

In particular, in the euclidean case, we recover the usual expression

$$\Delta f = -\sum_j \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j} f.$$

Let $f \in C^2(M)$ and $h \in C^1(M)$ such that hdf has compact support on M. Then we have Green's Formula

$$(\Delta f, h) = \int_M < df, dh > dV - \int_{\partial M} h\frac{df}{dn} dA$$

where $\frac{df}{dn}$ denote the derivative of f in the direction of the outward unit normal vector field n on ∂M and dA the volume form on ∂M.

In particular, if one of the following conditions $\partial M = \emptyset$, $h_{|\partial M} = 0$ or $(\frac{df}{dn})_{|\partial M} = 0$, then we have the very useful relation

$$(\Delta f, h) = (df, dh).$$

In the sequel, we will study the following eigenvalue problems when $\bar{M}$ is compact :

– Closed Problem :

$$\Delta f = \lambda f\ in\ M;\ \partial M = \emptyset;$$

– Dirichlet Problem

$$\Delta f = \lambda f\ in\ M;\ f_{|\partial M = 0};$$

– Neumann Problem :

$$\Delta f = \lambda f \ in \ M; \ (\frac{df}{dn})_{|\partial M} = 0.$$

We have the following standard result about the spectrum, see [Be]

Theorem 1. *Let M be a compact manifold with boundary ∂M (eventually empty), and consider one of the above eigenvalue problem. Then :*

1. *The set of eigenvalue consists of an infinite sequence $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq ... \to \infty$, where 0 is not an eigenvalue in the Dirichlet problem ;*
2. *Each eigenvalue has finite multiplicity and the eigenspaces corresponding to distinct eigenvalues are $L^2(M)$-orthogonal ;*
3. *The direct sum of the eigenspaces $E(\lambda_i)$ is dense in $L^2(M)$ for the L^2-norm. Futhermore, each eigenfunction is C^∞-smooth and analytic.*

Remark 2. *The Laplace operator depends only on the given Riemannian metric. If*

$$F : (M, g) \to (N, h)$$

is an isometry, then (M, g) and (N, h) have the same spectrum, and if f is an eigenfunction on (N, h), then $F \circ f$ is an eigenfunction on (M, g) for the same eigenvalue.

It turns out that in general, the spectrum cannot be computed explicitly. The very few exceptions are manifolds like round sphere, flat tori, ball. In general, it is only possible to get estimate of the spectrum, and these estimation are related to the geometry of the manifold (M, g) we consider.

The Dirichlet problem has a physical interpretation in terms of "how sounds a drum", see Courant-Hilbert. And this give a good intuition of the two types of problems we are dealing with : there is clearly a relation between the shape of a drum and how it sounds. The shape is the geometry of the manifold, that is all the Riemannian invariants depending on the Riemannian metric g : curvature, diameter, volume, injectivity radius, systole, etc... How the drum sounds corresponds to the spectrum of the manifold.

– The "direct" problem : we see a drum and try to imagine how it sounds. The mathematical translation is : some of the geometry of the manifold is given, in general in term of bound on the Riemannian invariants. Then, estimate part of the spectrum with respect to these bounds.
– The "inverse" problem : we hear a drum without seeing it and try to deduce informations on its shape. The mathematical question associate to this is to decide if the spectrum determines the geometry. The most famous question is about the isospectrality : if two Riemannian manifolds have the same spectrum, are they isometric ?

In these notes, I will focus only in the direct problem (for an introduction to the inverse problem, see the survey of C. Gordon, [Go]) and indeed on a very particular part of it : the main problem I will investigate is "can λ_1 be very large or very small ?". The question seems trivial or naive at the first view, but it is not, and I will try to explain that partial answer to it are closely related to geometric but also topological properties of the considered Riemannian manifold.

Of course, there is a trivial way to produce arbitrarily small or large eigenvalues : for any constant $c > 0$, $\lambda_k(c^2 g) = \frac{1}{c^2}\lambda_k(g)$. So, we have to introduce some normalizations, in order to avoid the trivial deformation of the metric given by an homothety. Most of the time, these normalizations are of the type "Volume is constant" or "curvature and diameter are bounded".

To investigate the Laplace equation $\Delta f = \lambda f$ is a priori a problem of analysis. To introduce some geometry on it, it is very relevant to look at the variational characterization of the spectrum. To this aim, let us introduce the Rayleigh quotient. If a function f lies in $H^1(M)$ in the closed and Neumann problem, and on $H^1_0(M)$ in the Dirichlet problem, the Rayleigh quotient of f is

$$R(f) = \frac{\int_M |df|^2 dV}{\int_M f^2 dV} = \frac{(df, df)}{(f, f)}.$$

Theorem 3. *(Variational characterization of the spectrum) Let us consider on of the 3 eigenvalues problems. We denote by $\{f_i\}$ an orthonormal system of eigenfunctions associated to the eigenvalues $\{\lambda_i\}$.*

1. $$\lambda_k = inf\{R(u) : u \neq 0; u \perp f_0, .., f_{k-1}\}$$

   where $u \in H^1(M)$ ($u \in H^1_0(M)$ for the Dirichlet eigenvalue problem) and $R(u) = \lambda_k$ if and only if u is an eigenfunction for λ_k.

   At view of this variational characterization, we can think we have to know the first k or $k-1$-eigenfunctions to estimate λ_k : this is not the case :

2. *Max-Min : we have*

   $$\lambda_k = sup_{V_k} inf\{R(u) : u \neq 0,\ u \perp V_k\}$$

   where V_k runs through k-dimensional subspaces of $H^1(M)$ or $C^\infty(M)$ ($(k-1)$-dimensional subspaces of $H^1_0(M)$ for the Dirichlet eigenvalue problem).

3. *Min-Max : we have*

   $$\lambda_k = inf_{V_k} sup\{R(u) : u \neq 0,\ u \in V_k\}$$

where V_k runs through $k+1$-dimensional subspaces of $H^1(M)$ or $C^\infty(M)$ (k-dimensional subspaces of $H^1_0(M)$ for the Dirichlet eigenvalue problem).

Remark 4. *We can see already two advantages to this variational characterisation of the spectrum. First, we see that we have not to work with solutions of the Laplace equation, but only with "test functions", which is easier. Then, we have only to control one derivative of the test function, and not two, as in the case of the Laplace equation.*

To see this concretely, let us give a couple of simple examples.

Example 5. ***Monotonicity in the Dirichlet problem.*** *Let $\Omega_1 \subset \Omega_2 \subset (M, g)$, two domains of the same dimension n of a Riemannian manifold (M, g). Let us suppose that Ω_1 and Ω_2 are both compact connected manifolds with boundary. If we consider the Dirichlet eigenvalue problem for Ω_1 and Ω_2 with the induced metric, then for each k*

$$\lambda_k(\Omega_2) \leq \lambda_k(\Omega_1)$$

with equality if and only if $\Omega_1 = \Omega_2$.

The proof is very simple : each eigenfunction of Ω_1 may be extended by 0 on Ω_2 and may be used as a test function for the Dirichlet problem on Ω_2. So, we have already the inequality. In the equality case, the test function becomes an eigenfunction : because it is analytic, it can not be 0 on an open set, and $\Omega_1 = \Omega_2$.

Corollary 6. *If M is a compact manifold without boundary, and if Ω_1,...,Ω_{k+1} are domains in M with disjoint interiors, then*

$$\lambda_k(M, g) \leq max(\lambda_1(\Omega_1), ..., \lambda_1(\Omega_{k+1}))$$

The second example explains how to produce arbitrarily small eigenvalues for Riemannian manifold with fixed volume. This example is again very simple, but the same type of questions for the spectrum of the Laplace operator acting on p-forms is much less easy.

Example 7. ***The Cheeger's dumm-bell.*** *We explain this example for a domain in $\mathbb{R}^n$ but it is easily generalized as Riemannian manifold.*

The idea is to consider two balls of fixed volume V related by a small cylinder C of length $2L$ and radius ϵ. The first nonzero eigenvalue of the Neumann problem converge to 0 as the radius of the cylinder goes to 0. It is even possible to estimate very precisely the asymptotic of λ_1 in term of ϵ (see [An]), but here, we just shows that it converge to 0.

We choose a function f which value 1 on the first ball, -1 on the second, and decreasing linearly, so that le norm of its gradient is $\frac{2}{L}$. By construction we have $\int f = 0$, so that we have $\lambda_1 \leq R(f)$.

But the Rayleigh quotient is bounded above by

$$\frac{4VolC/L^2}{2V}$$

which goes to 0 *as* ϵ *does.*

A classical way to estimate the spectrum from below is to cut a manifold into small parts, where we have a reasonable knowledge of the spectrum, and try to get from this a control of the whole spectrum. As example, consider a compact Riemannian manifold M, and let $\Omega_1, ..., \Omega_m \subset M$ pairwise disjoint domains such that

$$M = \bar{\Omega_1} \cup ... \cup \bar{\Omega_m}.$$

Then, consider the Neumann boundary conditions on the Ω_k and arrange all the eigenvalues of $\Omega_1, ..., \Omega_m$ in increasing order, with repetition according to multiplicity

$$0 \leq \mu_1 \leq \mu_2 \leq ...$$

Then,

Proposition 8. *With the above notations, we have*

$$\lambda_k(M) \geq \mu_{k+1}.$$

Observe that if we have m domains, then $\mu_1 = ... = \mu_m = 0$, so that we get an effective estimate only for the m-th eigenvalue λ_m.

In order to show this proposition, we denote by $\{f_i\}$ a set or orthonormal eigenfunctions for the λ_i and by $\{\phi_i\}$ a set or orthonormal eigenfunctions for the $\{\mu_i\}$. Then consider the application

$$\psi : \mathbb{R}^{k+1} \to \mathbb{R}^k$$

given by

$$\psi(a_0, ..., a_k) = \{\int_{\Omega_{i_j}} f\phi_j\}_{j=1}^k$$

where $f = \sum_{r=0}^k a_r f_r$ and i_j is such that ϕ_j is defined on Ω_{i_j}.

The application ψ is linear, and has a non trivial kernel. There exists nonzero $(a_0, ..., a_k)$ with $f = \sum_{r=0}^k a_r f_r$ is orthogonal to $\phi_1, ..., \phi_k$, so that we have $R(f) \geq \mu_{k+1}$.

But, we have also $R(f) \leq \lambda_k$, so that we conclude that

$$\lambda_k \geq \mu_{k+1}.$$

Let us finish this introduction to the Laplace operator on functions by giving two typical results which show how the geometry allows to control the first nonzero eigenvalue in the closed eigenvalue problem.

The first one is the Cheeger's inequality, which generalizes in some sense the dumm-bell example. We present it in the case of a compact Riemannian manifold without boundary, but it may be generalized to compact manifolds with boundary (for both Neumann or Dirichlet boundary conditions) or to noncompact, complete, Riemannian manifolds.

Definition 9. *Let (M, g) be an n-dimensional compact Riemannian manifold without boundary. The Cheeger's isoperimetric constant $h = h(M)$ is defined as follows*

$$h(M) = inf_C \ J(C); \ J(C) = \frac{Vol_{n-1}C}{min(Vol_n M_1, Vol_n M_2)},$$

where C runs through all compact codimension one submanifolds, which divide M into two disjoint connected open submanifolds M_1, M_2 with common boundary $C = \partial M_1 = \partial M_2$.

Theorem 10. *Cheeger's inequality. We have the inequality*

$$\lambda_1(M, g) \geq \frac{h^2}{4}.$$

A proof may be found in Chavel's book [Ch] and developments and other statement in Buser's paper [Bu1]. This result is remarkable, because it relates an analytic quantity (λ_1) to a geometric quantity (h) without any other assumption on the geometry of the manifold. It turns out that an upper bound in term of the Cheeger's constant may be given, but under some geometrical assumptions : this is the Theorem of P. Buser (see [Bu2])

Theorem 11. *If $Ric(M, g) \geq -(n-1)^2\delta^2 g$, then,*

$$\lambda_1(M, g) \leq c(n)(|\delta| h(M, g) + h^2(M, g)).$$

A difficulty with the Cheeger's inequality is precisely to estimate the Cheeger's constant. In some situations, the result has to be understand in the sense where λ_1 may be used in order to estimate h.

The second one's gives an universal lower bound in term of the geometry of the manifold. Again, we state the result for a manifold without boundary, but other statements for manifolds with boundary may be found in [LY].

Theorem 12. *(See [LY]). Let (M, g) be a compact n-dimensional Riemannian manifold without boundary. Suppose that the Ricci curvature satisfies $Ric(M, g) \geq (n-1)K$ and that d denote the diameter of (M, g).*

Then,

$$\lambda_1(M,g) \geq \frac{exp - (1 + (1 - 4(n-1)^2 d^2 K)^{1/2})}{2(n-1)^2 d^2}.$$

1.2 The case of p-forms

Let us begin with the case of a compact oriented Riemannian manifold of dimension n without boundary. Denote by

$$\wedge^*(M) = \oplus_{p=0}^n \wedge^p (M)$$

the exterior algebra of M.

Denote by

$$d : \wedge^p(M) \to \wedge^{p+1}(M)$$

the exterior derivative.

There exists a linear operator $*$ (the Hodge-Star operator) which assigns to each p-form $\omega \in \wedge^p(M)$ an $(n-p)$-form $*\omega \in \wedge^{n-p}(M)$ and satisfies

$$** = (-1)^{p(n-p)}.$$

The codifferential operator is defined by

$$\delta = (-1)^{n(p+1)+1} * d*$$

and the Laplace-Beltrami operator $\Delta : \wedge^p(M) \to \wedge^p(M)$ (the Laplacian acting on p-diffenretial forms) by

$$\Delta = d\delta + \delta d.$$

Note that δ is 0 when applied to functions, so that the Laplacian acting on function reduce to $\Delta = \delta d$ on $\wedge^0(M)$.

For two smooth forms $\alpha, \beta \in \wedge^p(M)$, we define their L^2-scalar product by

$$(\alpha, \beta) = \int_M \alpha \wedge *\beta.$$

With respect to this scalar product, δ is the adjoint of d, and Δ is a symmetric operator on $\wedge^p(M)$. We have a similar result as for the Laplacian on functions (see [Ro])

Theorem 13. *Let (M,g) be a compact Riemannian manifold without boundary. Then, $L^2(C^\infty \wedge^p (M))$ has an orthonormal basis consisting of eigenforms of the Laplacian on p-forms. One can order the eigenforms so that the corresponding eigenvalues $\lambda_{k,p}$ satisfy*

$$0 < \lambda_{1,p} \leq \lambda_{2,p} \leq \lambda_{3,p} \leq ... \to \infty$$

The eigenvalues are positive, accumulate only at infinity and have finite multiplicity.

Let $H^p(M,g) = \{\omega \in C^\infty \wedge^p (M) : \Delta\omega = 0\}$ be the space of harmonic p-forms. Then $\Delta\omega = 0$ if and only if $d\omega = \delta\omega = 0$.

Theorem 14. ***(Hodge decomposition Theorem)*** *For each integer p with $0 \leq p \leq n$, H^p is finite dimensional and we denote $dimH^p(M)$ by $b_p(M)$, the p-th Betti number of M. Moreover, we have the following orthogonal direct sum decomposition of the space $C^\infty(\wedge^p(M))$:*

$$C^\infty(\wedge^p(M)) = d(C^\infty \wedge^{p-1} (M)) \oplus \delta(C^\infty \wedge^{p+1} (M)) \oplus H^p(M).$$

As in the case of functions, we have a variational characterization of the spectrum of p-forms. To this aim, let us introduce the Rayleigh quotient of $\omega \in C^\infty \wedge^p M$: it is given by

$$R(\omega) = \frac{\int_M (|d\omega|^2 + |\delta\omega|^2) dVol}{\int_M |\omega|^2 dVol}. \tag{1}$$

Then, we have

Theorem 15.

$$\lambda_{k,p}(M,g) = \underset{E}{min}\ max\{R(\omega) : \omega \in E\}$$

where E runs through all vector subspaces of dim $k + b_p(M)$ of $C^\infty \wedge^p M$.

The differential d and the codifferential δ commute to the Laplacian. It mean that if ω is a p-eigenform of Δ, then $d\omega$ is a $(p+1)$-eigenform and $\delta\omega$ is a $(p-1)$-eigenform (eventually 0). In fact, because of the Hodge decomposition, any (non harmonic) eigenform may be decomposed as a sum of exact and coexact eigenforms. If ω is an exact eigenform, $*\omega$ is a coexact eigenform.

If $\lambda > 0$, denote by $E'_p(\lambda)$ and $E''_p(\lambda)$ the eigenspaces of λ consisting respectively of exact and coexact p-forms. Then $d : E''_p(\lambda) \to E'_{p+1}$ and $\delta : E'_p(\lambda) \to E_{p-1}(\lambda)$ are vector spaces isomorphisms.

In particular, there is a copy of the spectrum of function in the 1-forms spectrum.

It is useful to have a min-max only for exact or coexact p-forms. For example, if we write

$$0 < \lambda''_{1,p} \leq \lambda'_{2,p} \leq \lambda'_{3,p} \leq ... \to \infty$$

the spectrum of coexact p-forms, we have

Theorem 16.

$$\lambda''_{k,p}(M,g) = \underset{E}{min}\ max\{R(\omega) : \omega \in E\}$$

where E runs through all vector subspaces of dim k of $\delta(C^\infty \wedge^{p-1} M)$.

We can ask for the Laplacian acting on p-forms the same questions as for the Laplacian on functions, that is to relate the spectrum and the geometry of the Riemannian manifold. However, there is a new fact in the study of the p-forms spectrum : it is that the role of the *topology* of the manifold is much more consequent. We can see this already for the harmonic forms, which are related to the de Rham cohomology, and we will develop this in the next section.

1.3 The De Rham Theory

Let M be an n-dimensional differentiable manifold. The exterior derivative d maps $C^\infty \wedge^p (M)$ to $C^\infty \wedge^{p+1} (M)$ and satisfies $d^2 = 0$. It induces a differential complex, the De Rham complex,

$$0 \to C^\infty \wedge^0 (M) \xrightarrow{d} C^\infty \wedge^1 (M) \xrightarrow{d} ... \xrightarrow{d} C^\infty \wedge^{n-1} (M) \xrightarrow{d} C^\infty \wedge^n (M) \xrightarrow{d} 0.$$

The cohomology of this complex is called the De Rham cohomology and the p'th cohomology group of the manifold M is given by

$$H^p_{dR}(M) = \frac{\{\omega \in C^\infty \wedge^p (M) : d\omega = 0\}}{\{d\theta : \theta \in C^\infty \wedge^{p-1} (M)\}}.$$

The dimension of $H^p_{dR}(M)$ is called the p'th Betti number of the manifold M and is denoted by $b_p(M)$. Since the spaces of closed and exact forms are both of infinite dimension, it is a nontrivial fact that, as M is compact, $b_p(M) < \infty$. We can define a pairing

$$H^p_{dR}(M) \times H^{n-p}_{dR}(M) \to \mathbb{R}$$

by

$$([\alpha], [\beta]) \mapsto \int_M \alpha \wedge \beta \tag{2}$$

where α and β are closed forms representing the cohomology class $[\alpha] \in H^p_{dR}(M)$, $[\beta] \in H^{n-p}_{dR}(M)$. We have

Theorem 17. ***Poincaré duality*** *The bilinear function (2) is a nonsingular pairing and determines an isomorphism of $H^{n-p}_{dR}(M)$ with the dual of $H^p_{dR}(M)$.*

A priori, it seems that the De Rham cohomology depends on the differential structure of the manifold M. In fact, it depends only on the topology : this is precisely the meaning of the De Rham Theorem, that is the existence of an isomorphism

$$r : H^p_{dR}(M) \to H^p_{sing}(M, \mathbb{R})$$

induced by

$$\omega \mapsto (\sigma \to \int_\sigma \omega)$$

for any p-form ω and p-chain σ, where $H^p_{sing}(M, \mathbb{R})$ denote the singular cohomology group of M with real coefficients.

Theorem 18. *The cohomology of the de Rham complex of M is isomorphic to the singular cohomology of M with real coefficients.*

Since every harmonic form on a compact Riemannian manifold (M, g) is closed, we have a map

$$h : Ker\Delta^p \to H^p_{dR}(M),\ \omega \mapsto H^p_{dR}(M).$$

With this, we see another aspect of the Hodge's Theorem

Theorem 19. ***(Hodge)*** *Let (M, g) be a compact, oriented manifold. Then*

$$Ker\Delta^p \tilde{=} H^p_{dR}(M).$$

So, through the multiplicity of the harmonic forms, we see that already that the topology of the underling manifold M is present. So a general question we can address in studying the first eigenvalues of the p-forms spectrum is the following : is it possible to "read" and to "measure" the importance and the influence of the topology of M in the frist nonzero eigenvalues ? One of the goal of the next lectures will precisely be to give some elements of answer to this question.

1.4 A basic example

The goal of this paragraph is to give a list of examples to get an intuition about the p-spectrum, and to see some differences with the case of functions.

Example 20. ***The spectrum of a product.*** *Let M and N be two compact Riemannian manifolds. If $\alpha \in C^\infty \wedge^p M$, $\beta \in C^\infty \wedge^q N$, then, considering the Riemannian product $M \times N$, a direct calculation shows that*

$$\Delta(\alpha \wedge \beta) = \Delta\alpha \wedge \beta + \alpha \wedge \Delta\beta.$$

This implies (see [Ta] p.356), that we get $spec^p(M \times N)$ by summing s-eigenvalues of M and r-eigenvalues of N with $r + s = p$.

In particular, we have the Künneth formula

$$H^p(M \times N) = \bigoplus_{r+s=p} H^r(M) \otimes H^s(N)$$

Example 21. *Let S^1_L be the circle of length L and (N,h) an $(n-1)$-dimensional compact Riemannian manifold. We consider the product $M = S^1_L \times N$ with the product metric. We want to study the spectrum as N is fixed and as $L \to \infty$.*

Recall that the first nonzero eigenvalue of S^1_L is equal to $\frac{4\pi^2}{L^2}$ and goes to 0 as $L \to \infty$. We already deduce from this that

$$\lambda_{1,0}(S^1_L \times N) = \frac{4\pi^2}{L^2}$$

and goes to 0 as $L \to \infty$. So the same is true for $\lambda_{1,1}(S^1_L \times N)$, $\lambda_{1,n}(S^1_L \times N)$ and $\lambda_{1,n-1}(S^1_L \times N)$.

The situation may be different for $\lambda_{1,p}$, $2 \leq p \leq n-2$. To see this, il suffices to consider two examples : the first one is to take $N = S^{n-1}$, the $(n-1)$-dimensional sphere (with the canonical metric). As S^{n-1} has no harmonic p-forms if $1 \leq p \leq n-2$, then

$$\lambda_{1,p}(S^1_L \times S^{n-1}) \geq C > 0$$

independently of L if $2 \leq p \leq n-2$, where $C = min\{\lambda_{1,1}(S^{n-1}), ..., \lambda_{1,n-2}(S^{n-1})\}$.

So, the situation differs drastically as in the case of functions.

In the second example, we take $N = T^{n-1}$, where T^{n-1} is the $(n-1)$-dimensional flat torus $S^1 \times ... \times S^1$. In this case, we have for each $0 \leq p \leq n$,

$$\lambda_{1,p}(S^1_L \times T^{n-1}) = \frac{4\pi^2}{L^2}$$

as $L \to \infty$.

So, the situation differs drastically from the first example.

The explanation is that the behaviour of $\lambda_{1,p}(M)$ depends strongly in this case from the topology of the manifold N in the product, and more precisely from its cohomology. It there is a harmonic form α of degree k, then we can "lift" the spectrum of S^1_L and associate to an eigenfunction f the k-form $f \wedge \alpha$ (or the (k+1)-forms $f dt \wedge \alpha$) which is an eigenform with the same eigenvalue.

With these two examples, we see that two Riemannian manifolds with bounded curvature and injectivity radius, which are Hausdorff close, may have spectrum which are not comparable. This differs completely of the intuition we have in the case of function (see Mantuano [Ma] for a precise statement).

Remark 22. *Note that in the previous example, the sectional curvature satisfies $K(S^1_L \times S^{n-1}) \geq 0$. So we have a family of compact Riemannian manifolds of nonnegative sectional curvature, whose diameter goes to ∞ and $\lambda_{1,p}$ is uniformly bounded below by a strictly positive constant as $2 \leq p \leq n-2$. This contrast with a general result of Cheeger and of Cheng in the case of functions.*

2 Lecture 2 : Some examples and results for the p-forms spectrum

In the sequel, we will generalize a lot the ideas contained in the example of the last lecture. The first generalization of a product is to consider a fiber bundle. It turns out that this point of view is very powerful. In this section devoted to examples, I will look at a special case, where it is easy to do explicit calculations and which gives a very good idea of the different problems and questions we can meet.

2.1 The p-spectrum of the Berger's spheres and of the dummbell

Example 23. *In this example, we will study the spectrum of a family of Riemannian metrics on the sphere of odd dimension $2n+1$, but we will see it as an S^1-bundle on the complex projective space $\mathbb{C}P^n$. The details of the construction and of the calculations are in [CC1].*

We define

$$S^{2n+1} = \{(z_1, ..., z_{n+1} \in \mathbb{C}^{n+1} : |z_1|^2 + ... + |z_{n+1}|^2 = 1\}.$$

There is an isometric action of S^1 on S^{2n+1} given by

$$(e^{i\theta}, (z_1, ..., z_{n+1})) \mapsto (e^{i\theta} z_1, ..., e^{i\theta} z_{n+1}).$$

The complex projective space $\mathbb{C}P^n$ is the quotient of S^{2n+1} by this action, and this allow to see S^{2n+1} as a S^1-principal bundle

$$S^1 \hookrightarrow S^{2n+1} \xrightarrow{\pi} \mathbb{C}P^n,$$

where the canonical projection π is a Riemannian submersion.

(Note that I will not use very strongly the formalism of vector bundle in these notes.)

The connection form of the S^1-bundle is represented by $\omega = X^\sharp$, where X is the unit vector field tangent to the action of S^1. In this particular case, it is not so difficult to calculate $d\omega$.

The derivative $d\omega$ is horizontal : this comes from the classical formula

$$d\omega(U, V) = U\omega(V) - V\omega(U) - \omega([U, V])$$

for two vector fields U and V.

If $U = X$ and V is horizontal, that is orthogonal to X in each point, a property of the Lie bracket make that $[X, V] = 0$ so that

$$d\omega(X, V) = 0$$

because $\omega(X) = 1$ is a constant function and $\omega(V) = 0$.

If U, V are horizontal, then

$$d\omega(U, V) = -\omega([U, V])$$

that is

$$d\omega = \pi^*(\Omega)$$

where Ω is the curvature form of the bundle, but also the Kähler form of the basis $\mathbb{C}P^n$.

Now, the idea is to use the so called "canonical variation of the metric" (or to introduce the so called "Berger'sphere"). It consists in introducing a family of Riemannian metric on S^{2n+1} obtained by deforming the canonical one's. Denote by H_p the horizontal space at the point p (that is the subspace of T_pS^{2n+1} orthogonal to $X(p)$). Then, define

$$g_t(p)(U, V) = \begin{cases} t^2 g(p)(X, X) \ if \ U = V = X \\ g(p)(U, V) \ if \ U, V \in H_p \\ 0 \ if \ U = X, V \in H_p \end{cases}$$

and extend it linearly.

We can continue the calculation of $\Delta\omega$ for the Riemannian metric g_t.

Recall that $d\omega = \pi^*\Omega$, where Ω is the Kähler form of $\mathbb{C}P^n$. So, $*\pi^*\Omega$ is a multiple of $X^\sharp \wedge \pi^*(*\Omega)$, and the multiple has to be choose such that the forms $\pi^*\Omega$ and $*\pi^*\Omega$ have the same norm with respect to g_t.

As the norm of $X^\sharp$ is now $\frac{1}{t}$, the multiple has to be t, so that we can write

$$*\pi^*\Omega = tX^\sharp \wedge \pi^*(*\Omega).$$

As we apply d again, we can use exactly the same calculation as before, because

$$d\pi^*(*\Omega) = \pi^*(d * \Omega) = 0$$

because of the harmonicity of Ω.

At the end, we get

$$\delta d\omega = t^2\omega.$$

On the other hand, we have $*\omega = \pi^*(\Omega_0)$, where Ω_0 is the volume form of $\mathbb{C}P^n$, so that $d * \omega = 0$.

At the end, we get

$$\Delta\omega = t^2\omega.$$

Moreover, the same calculation show the following : take any $2k$ form $\alpha = \Omega \wedge ... \wedge \Omega$ of the basis $\mathbb{C}P^n$, then $\pi^*(\alpha)$ and $\omega \wedge \pi^*(\alpha)$ are respectively a $2k$ and a $(2k+1)$-eigenform with respect to g_t for the eigenvalue t^2.

A classical calculation shows also that as $t \to 0$, the sectional curvature of (S^{2n+1}, g_t) stay bounded.

So, we are in position to state some very interesting consequences of the two examples.

1. In the Example 21, we get (by shrinking the sphere S^{n-1}) a family of Riemannian positively curved manifolds, with diameter $\to \infty$ and $\lambda_{1,p}$ arbitrarily large for $2 \leq p \leq n-2$;
2. In Example 23, we have a family of Riemannian manifolds with bounded sectional curvature and diameter, and $\lambda_{1,p} \to 0$ for $1 \leq p \leq 2n$: this contrast with the case of functions (see Theorem 12) and will lead to a lot of questions.
3. It is also possible to understand Example 23 as follow : as $t \to 0$, the family (S^{2n+1}, g_t) converge in some sense (Gromov-Hausdorff convergence) to the basis $\mathbb{C}P^n$ of the bundle. But if S^{2n+1} has nonzero cohomology only in dimension 0 and $2n+1$, $\mathbb{C}P^n$ has non vanishing cohomology in all even degree. And the harmonic form associated to this cohomology will give a small eigenvalue for form as lifted to the sphere.

Example 24. ***The dumm-bell revisited*** *The goal of this Example is just to show that the intuition given by the dumm-bell in the case of function is no more true in the case of forms. The details of the construction and of the result I describe are rather technical and may be find in [AC].*

Let us consider two compact Riemannian manifolds M_1 and M_2 of dimension n. We make a small hole and we join M_1 and M_2 by a thin cylinder isometric to $[O, L] \times S^{n-1}_\epsilon$ where S^{n-1}_ϵ is the round $(n-1)$-sphere of radius ϵ. Let M_ϵ denote the manifold we get with this construction. The question address in [AC2] is to study the evolution of the p-spectrum of M_ϵ as $\epsilon \to 0$.

As $1 < p < n-1$, $dimH^p(M_\epsilon) = dimH^p(M_1) + dimH^p(M_2)$, as we can see by a Mayer-Vietoris argument.

In [AC], we proved that, if $1 < p < n-1$, the p-spectrum of M_ϵ converge to the union of p-spec(M_1) and p-spec(M_2) as $\epsilon \to 0$.

It was already known from previous works of C. Anné that, as $\epsilon \to 0$, the spectrum of functions of M_ϵ converge to the union of both 0-spectrum of M_1 and M_2, with the Dirichlet spectrum of the interval $[0, L]$. From this, we can also deduce the behaviour of the 1-spectrum.

Again, it is impossible to explain in a few lines this very technical result, but it is possible to get an intuition if we say a few word of the easier part of the problem, that is how to construct test forms on M_ϵ from eigenforms of M_1 and M_2.

As we do a small hole on a manifold N, it is straightforward (see [AC1]) to replace an eigenform α on N by a form β equal to 0 near the hole and to α outside a neighbourhood

of the hole, without affecting to much the Rayleigh quotient. Then, as we do this for M_1 and M_2, we can extend the test form by 0 on the cylinder, and get a test form on M_ϵ. This give a majoration of the spectrum of M_ϵ by the union of the spectrum of M_1 and M_2.

Where a difference appears between functions and p-forms, $(1 < p < n-1)$, is that we can try to construct test forms with the cylinder. This is possible for functions (and 1-forms) by considering the Dirichlet eigenfunctions of the cylinder and extending them by 0 on M_ϵ. If we do the same for p-forms, $(1 < p < n-1)$, the Rayleigh quotient on the cylinder go to ∞ as $\epsilon \to 0$ as explained in Example 21, and in fact we can show that the eigenforms of M_ϵ tend to concentrate on both M_1 and M_2, but not on the cylinder.

2.2 Lower bound for the p-spectrum

The goal of this section is to give a first theorem allowing to estimate the p-spectrum. Without surprise, it turns out that this theorem are closely related to the cohomology of the manifold we consider, and that the methods of proofs are closed to methods we use in cohomology, as the Mayer-Vietoris sequence and the proof of the De Rham Theorem.

We have seen in Example 23 that a control of the sectional curvature and of the diameter in not enough to guarantee a lower bound for $\lambda_{1,p}$: we can think that we can get such a lower bound if we add a control of the injectivity radius (or equivalently of the volume) and this is indeed true.

It is quite easy to get such a result without having an explicit lower bound depending on the geometry, but only its existence, and it was done in [CC1], but it is much more difficult to have an *explicit* lower bound, and this is partially done in the paper of Chanillo and Trèves [CT].

We give here a statement which differs a little of the statement of the paper (which is a little more general)

Theorem 25. ***(Chanillo-Trèves)*** *Let (M, g) be an n-dimensional compact Riemannian manifold such that its sectional curvature, diameter and injectivity radius satisfies respectively $|K(M, g)| \leq a$, $diam(M, g) \leq d$ and $Inj(M, g) \geq r > 0$. Then, there exists a constant $c(n, a, d) >$ such that*

$$\lambda_{1,p}(M, g) \geq c(n, a, d) r^{4n^2+8n-2}.$$

The constant $c(n, a, d)$ is not given explicitly, but this may be done by a carefull reading of the proof.

This Theorem is at the origins of some very nice questions. The first one is very general and will be develop in the section 4 : Let us consider the set $\mathcal{M}(n, a, d)$ of n-dimensional Riemannian manifold with $|K(M, g)| \leq a$ and $diam(M, g) \leq d$. If a family

$M_i \in \mathcal{M}(n,a,d)$ is such that for one p, $\lambda_{p,1}(M_i) \to 0$, then Theorem 25 implies that $Inj(M_i) \to 0$ (or on an equivalent way, $Vol(M_i) \to 0$. So, there is the converse question : If a family $M_i \in \mathcal{M}(n,a,d)$ is such that $Inj(M_i) \to 0$ (or $Vol(M_i) \to 0$), then does it gives small eigenvalues, how many, and how are they related to $Inj(M_i)$ or $Vol(M_i)$, to the topology, etc.. Now a lot is known (and also a lot is unknow) about this question (see section 4).

There are also some open questions that we can ask already

Open question 1. *Is it possible to get an analog statement as Theorem 25 with a bound only on the Ricci curvature ?*

Open question 2. *In Theorem 25, the bounds are not optimal. Is it possible to find the optimal bound in term of Inj or Vol, and, if not, at least to do much better ?*

In the next lecture, we will establish another result coming from the PHD thesis of J. Mc Gowan (see [MG]). It is not a good idea to give the most general formulation of this result, because it would be very complicated to read. We will give two statements in particular cases, the statement of the original paper of Mc Gowan and the statement we used to estimate specifically $\lambda_{1,p}$ (see for example [Gu], [GP]).

The idea behind this Theorem is that, in order to estimate the spectrum of a manifold, it may be convenient to cut it into parts that we know, and then to estimate the whole spectrum thanks to the spectrum of the parts. This approach leads us to consider problem with boundary. So, we will explain what are the "right" boundary conditions for the Laplacian acting on p-forms. Then we will recall what we do in general in the case of functions, because this approach is well known is this context. After that we will state and explain the proof of Mc Gowan's Theorem.

Boundary conditions for forms. Dirichlet and Neumann boundary conditions on functions can be generalized to p-forms. Let M be a compact Riemannian manifold with boundary. We have a generalization of Green's formula (see [Ta] p. 361) for $\omega, \eta \in C^\infty \wedge^p M$:

$$(\Delta\omega, \eta) = (d\omega, d\eta) + (\delta\omega, \delta\eta) - \int_{\partial M} (< \delta\omega, i_\nu \eta > - < i_\nu(d\omega), \eta >)$$

where i_ν denotes the inner product with ν, and ν is the outward pointing normal vector.

We choose the boundary condition in order to have a classical Green's formula. The *absolute* boundary conditions are

$$i_\nu \omega = 0; \; i_\nu d\omega = 0 \tag{3}$$

and the *relative* boundary conditions are

$$\omega_{|\partial M} = 0; \; \delta\omega_{|\partial M} = 0 \tag{4}$$

Remark 26. *If ω satisfies the relative or absolute boundary conditions, then $\Delta\omega = 0$ implies $d\omega = \delta\omega = 0$.*

For both absolute and relative boundary conditions, the spectrum has the same properties as in the case without boundary. We can find the following result in [GLP] (Theorem 1.5.4, p. 37)

Theorem 27. *Let (M, g) be a compact Riemannian manifold with smooth boundary,and consider one of the eigenvalues problems with absolute (A) or relative (R) boundary conditions. Then, $L^2(C^\infty \wedge^p (M))$ has an orthonormal basis consisting of eigenforms of the Laplacian on p-forms with (A) or (R). One can order the eigenforms so that the corresponding eigenvalues $\lambda_{k,p}$ satisfy*

$$0 < \lambda_{1,p} \leq \lambda_{2,p} \leq \lambda_{3,p} \leq ... \to \infty$$

The eigenvalues are positive, accumulate only at infinity and have finite multiplicity.

Let $H^p(M, g) = \{\omega \in C^\infty \wedge^p (M) : \Delta\omega = 0\}$ be the space of harmonic p-forms. Then $\Delta\omega = 0$ if and only if $d\omega = \delta\omega = 0$.

Remark 28. *The Hodge operator $(*)$ interchanges the boundary conditions, so that there is a correspondence between the p-spectrum for absolute boundary conditions and $(n-p)$-spectrum for relative boundary conditions, where n is the dimension of M. So, in the sequel, I will focus on the case with absolute boundary conditions.*

As in the case without boundary, there is a relation between the p-harmonic forms and the topology of the underlying manifold M, and the natural injection from the harmonic forms in the De Rham cohomology gives an isomorphism, so that the dimension of the space $H^p(M, g)$ of p-harmonic forms (with absolute boundary condition) coincide with the dimension of the real De Rham cohomology of degree p of M.

We also have a Hodge decomposition (see [Ta], prop. 9.8, p.367)

Theorem 29. *If $\omega \in C^\infty(\wedge^p M)$, we have the orthogonal decomposition*

$$\omega = d\delta\alpha + \delta d\beta + \gamma \tag{5}$$

where α,β and γ are of class C^∞ and satisfies absolute boundary conditions.

Note in particular that, moreover, $d\beta$ satisfies $i_\nu(d\beta) = 0$. This implies (after calculations) a useful property we will use in the sequel, that is $i_\nu(\delta d\beta) = 0$.

Remark 30. *The fact, mentionned in Theorem 29 that, for a form ω, $i_\nu\omega = 0$ implies $i_\nu(\delta\omega) = 0$ is often used without proof. We give here a short proof of this fact communicated by A. Savo.*

We considere an orthonormal frame field $(e_1, ..., e_{n-1}, \nu)$ near a point of the boundary, where the e_k's are tangent to ∂M. If L denotes the second fundamental form of ∂M, we have

$$\nabla_{e_k}\nu = -\sum_{j=1}^{n-1} L(e_k, e_j)e_j.$$

If $X_1, ..., X_m$ are tangent vectors, we have

$$i_\nu \delta\omega(X_1, ..., X_m) = \delta\omega(\nu, X_1, ..., X_m) =$$

$$= -\sum_{k=1}^{n-1} \nabla_{e_k}\omega(e_k, \nu, X_1, ..., X_m) - \nabla_\nu\omega(\nu, \nu, X_1, ..., X_m) =$$

$$= -\sum_{k=1}^{n-1} e_k.\omega(e_k, \nu, X_1, ..., X_m) + \sum_{k=1}^{n-1} \omega(e_k, \nabla_{e_k}\nu, X_1, ..., X_m)$$

$$= -\sum_{j,k=1}^{n-1} L(e_j, e_k)\omega(e_k, e_j, \nu, X_1, ..., X_m),$$

which is zero because L is symetric and ω skew-symmetric.

We have a variational characterisation of the spectrum in this case

Theorem 31.

$$\lambda_{k,p}(M, g) = \underset{E}{min}\ max\{R(\omega) : \omega \in E\}$$

where E runs through all vector subspaces of dim $k + b_p(M)$ of $C^\infty \wedge^p M$ satisfying the boundary condition $i_\nu\omega = 0$.

Again, we can restrict to coexact p-forms, and get

Theorem 32.

$$\lambda''_{k,p}(M, g) = \underset{E}{min}\ max\{R(\omega) : \omega \in E\}$$

where E runs through all vector subspaces of dim k of $\delta C^\infty \wedge^p M$ satisfying the boundary condition $i_\nu\omega = 0$.

3 Lecture 3 : The Theorem of Mc Gowan : applications and proof of the theorem

3.1 Statement of the theorem

We can now state a version of J. Mc Gowan's Lemma (the version of Gentile-Pagliara).

Theorem 33. *Let M be a compact Riemannian manifold without boundary of dimension n and $\{U_i\}_{i=0}^k$ an open cover of M, such that there are no intersections of order higher than 2. Let $U_{ij} = U_i \cap U_j$. Suppose further, that $H^{p-1}(U_{ij}) = 0$ for all i, j. Denote by $\mu(U_i)$ (resp. $\mu(U_{ij})$) the smallest positive eigenvalue of the Laplacian acting on exact forms of degree p on U_i (resp. of degree $p-1$ on U_{ij}) satisfying absolute boundary conditions. Then the first nonzero eigenvalue $\lambda'_{1,p}(M)$ on exact p-forms satisfies*

$$\lambda'_{1,p}(M) \geq \frac{2^{-3}}{\sum_{i=1}^k (\frac{1}{\mu(U_i)} + \sum_{j=1}^{m_i} (\frac{w_{n,p} c_\rho}{\mu(U_{ij})} + 1)(\frac{1}{\mu(U_i)} + \frac{1}{\mu(U_j)}))} \tag{6}$$

where m_i is the number of j, $j \neq i$, for which $U_i \cap U_j \neq \emptyset$, $w_{n,p}$ a combinatorial constant which depends on p and n, $c_\rho = (max)_i (max_{x \in U_i}) |\nabla \rho_i(x)|^2$ for a fixed partition of unity $\{\rho_i\}_{i=1}^k$ subordinate to the given cover.

In order to get a lower bound on $\lambda_{1,p}(M)$, we have to control $\lambda'_{1,p}(M)$ and $\lambda'_{1,p+1}(M)$

Of course, the hypothesis of this Theorem seem to be very strong. We will give soon another statement of this result (in fact the original statement), but I will first explain how to use Theorem 39 to construct large eigenvalues for p-forms on any compact manifolds.

Example 34. *There is a quit large class of examples that the Gentile-Pagliara's version of Mc Gowan's Theorem allows to control. Consider a fundamental piece consisting on a compact Riemannian manifold with boundary (M, g_0). Moreover, suppose that the boundary ∂M consists 4 disjoint geodesic hypersurfaces isometric to the same sphere S^{n-1}. Around the bondary, we can also suppose that M, g_0 is isometric to a product $S^{n-1} \times I$.*

Then, it is possible to glue different pieces together, following, as example, a regular graph of degree 4 (see [CM] for detailled construction of this type). Then, an immediate application of the above theorem show that, for $2 \leq p \leq n-2$, the p-spectrum of such a compact manifold is controled from below by the spectrum of (M, g_0) with absolute boundary condition.

3.2 Construction of large eigenvalues for p-forms

Theorem 35. *Every compact, connected manifold M^n of dimension $n \geq 4$ admits metrics g of volume one with arbitrarily large $\lambda_{1,p}(g)$ for all $2 \leq p \leq n-2$.*

Moreover, if a Riemannian metric g_1 on M is given, we can choose g in the conformal class of g_1.

For motivations and history about this question, see the lecture of A. El Soufi, or the reference [Co].

Proof of Theorem 35 (For the details, see [GP]) We begin we a metric g_1 on M, and we deform this metric around a point, in order to "add a long cylinder" to the manifold M. Note that does not change the topology $of M$. It is like if we have a manifold (M_1, g_1) gluing with a cylinder $Z = I \times S^{n-1}$ closed by half a sphere H_1.

More precisely, we put a family g_t of Riemannian metric equal to g_1 on M_1, to $[0, t] \times S^{n-1}$ on Z and to the canonical metric of the half-sphere on H_1. Moreover, we identify ∂M_1 with ∂Z at $\{t\} \times S^n$ and ∂H_1 to $\{0\} \times S^{n-1}$.

We set $Z_1 = [0, 1[\times S^{n-1}$, $Z_2 =]0, t[$ and $Z_3 =]t-1, t] \times S^{n-1}$ and

$$U_1 = H_1 \cup Z_1;\ U_2 = Z_3 \cup M_1\ U_3 = Z_2.$$

So, U_1, U_2, U_3 is an open covering of M, and satisfies exactly the hypothesis of Theorem 39, because

$$U_1 \cap U_2 = \emptyset;\ U_1 \cap U_3 =]0, 1[\times S^n;\ U_2 \cap U_3 =]t-1, t[\times S^{n-1},$$

and $U_1 \cap U_3$, $U_2 \cap U_3$ have the cohomology of the sphere S^{n-1}, that is 0 in degree $2, ..., n-2$.

Theorem 39 will allow us to control the first nonzero eigenvalue of p-exact forms with the first nonzero eigenvalue of the absolute boundary problem for exact p-forms on U_1, U_2, U_3 and exact $(p-1)$-forms on $U_1 \cap U_3$ and $U_2 \cap U_3$.

The fundamental observation is now that as t varies, U_1, U_3, $U_1 \cap U_3$ and $U_2 \cap U_3$ are fixed and U_2 is depending on t, but in a very simple way, because $U_2 =]0, t[\times S^{n-1}$.

Exactly as in Example 21, we have

$$\mu_{1,p}(U_3) \geq C > 0$$

if $2 \leq p \leq n-1$, where C is independent of t. This allow to control the p-exact spectrum of M from below by a positive constant not depending on t, for $2 \leq p \leq n-1$, and as a consequence, $\lambda_{1,p}(M)$ for $2 \leq p \leq n-2$.

But, as $t \to \infty$, $Vol(M, g_t) \to \infty$. So, after renormalization to a volume 1 metric, $\lambda_{1,p} \to \infty$ with t.

It turn out that our construction of the cylinder is a conformal deformation of a euclidean metric. Now, if we begin with any metric g', we first replace it by a metric g'' flat around a point, do our construction. We show then that the same conformal deformation of the initial metric g' gives also large eigenvalue. The details are in [CE1] and [CE2].

Remark 36. *This type of construction is now used in a lot of different problems.*

Note that the construction of large eigenvalues for function is possible, but different. So, the case of 1-forms is very special, and we have the following

Open question 3. *Let M be a given compact manifold of dimension $n \geq 3$. Is it possible to construct on M a family of volume 1 Riemannian metric with arbitrarily large $\lambda_{1,1}$?*

A problem in Theorem 39 is of course that, in general, there is no reason for $U_i \cap U_j = \emptyset$. It turns out that in the initial statement of J. Mc Gowan, this was not suppose. But there is a price to pay : it is not possible to estimate the *first* nonzero eigenvalue, but only the N-th. The lemma was also stated for 1-forms, which do that it is not to difficult to read.

Theorem 37. *Let M be a compact Riemannian manifold without boundary of dimension n and $\{U_i\}_{i=0}^k$ an open cover of M.Let $U_{ij} = U_i \cap U_j$. Denote by $\mu(U_i)$ (resp. $\mu(U_{ij})$) the smallest positive eigenvalue of the Laplacian acting on exact forms of degree p on U_i (resp. of degree $p-1$ on U_{ij}) satisfying absolute boundary conditions. Let $N_1 = \sum_{i,j} dimH^1(U_{i,j})$, $N_2 = \sum_{i,j,l} dimH^0(U_i \cap U_j \cap U_l)$ and set $N = 1 + N_1 + N_2$.*

Then the N-th eigenvalue $\lambda'_{N,2}(M)$ on exact 2-forms satisfies

$$\lambda'_{N,2}(M) \geq \frac{1}{\sum_{i=1}^k (\frac{1}{\mu(U_i)} + \sum_{j=1}^{m_i} (\frac{w_{n,p} c_\rho}{\mu(U_{ij})} + 1)(\frac{1}{\mu(U_i)} + \frac{1}{\mu(U_j)}))} \tag{7}$$

where m_i is the number of j, $j \neq i$, for which $U_i \cap U_j \neq \emptyset$, $w_{n,p}$ a combinatorial constant which depends on p and n, $c_\rho = (max)_i (max_{x \in U_i}) |\nabla \rho_i(x)|^2$ for a fixed partition of unity $\{\rho_i\}_{i=1}^k$ subordinate to the given cover.

Remark 38.
1. *If one want to estimate eigenvalues for forms of higher degree, we have to take account the cohomology of higher degree of the intersection $U_{i_1} \cap ... \cap U_{i_r}$.*
2. *If we try to apply this Theorem on function (that is on 1-exact forms), we need to take account of the cohomology of degree 0 of the intersection. This cohomology is never 0, so that we never have informations on $\lambda_{1,0}$.*

3.3 Proof of Mc Gowan's Theorem

The goal is to prove the Gentile-Pagliara version of Mc Gowan's Theorem

Theorem 39. *Let M be a compact Riemannian manifold without boundary of dimension n and $\{U_i\}_{i=0}^k$ an open cover of M, such that there are no intersections of order higher than 2. Let $U_{ij} = U_i \cap U_j$. Suppose further, that $H^{p-1}(U_{ij}) = 0$ for all i, j. Denote by $\mu(U_i)$ (resp. $\mu(U_{ij})$) the smallest positive eigenvalue of the Laplacian acting on exact forms of degree p on U_i (resp. of degree $p-1$ on U_{ij}) satisfying absolute boundary conditions. Then the first nonzero eigenvalue $\lambda'_{1,p}(M)$ on exact p-forms satisfies*

$$\lambda'_{1,p}(M) \geq \frac{2^{-3}}{\sum_{i=1}^k (\frac{1}{\mu(U_i)} + \sum_{j=1}^{m_i} (\frac{w_{n,p} c_\rho}{\mu(U_{ij})} + 1)(\frac{1}{\mu(U_i)} + \frac{1}{\mu(U_j)}))} \tag{8}$$

where m_i is the number of j, $j \neq i$, for which $U_i \cap U_j \neq \emptyset$, $w_{n,p}$ a combinatorial constant which depends on p and n, $c_\rho = (max)_i(max_{x \in U_i})|\nabla \rho_i(x)|^2$ for a fixed partition of unity $\{\rho_i\}_{i=1}^k$ subordinate to the given cover.

The difficulty is that the restriction of an eigenform to an open subset does not, in general, satisfies the absolute boundary conditions. However, there is an approach of J. Dodziuk which allow to turn more or less this problem :

Theorem 40. *Let (M, g) be a compact Riemannian manifold with boundary. The spectrum of the Laplacian $0 < \lambda'_{1,p} \leq \lambda'_{2,p} \leq ...$, acting on exact p-forms which satisfies absolute boundary conditions can be computed by*

$$\lambda'_{k,p} = inf_{V_k} sup_{V_k - \{0\}} \{ \frac{(\phi, \phi)}{(\eta, \eta)} : d\eta = \phi \} \tag{9}$$

where V_k range over all dimension k subspaces of $C^\infty(\wedge^p M) \cap L^2(\wedge^p M)$ exact p-forms, and $\eta \in C^\infty(\wedge^{p-1} M) \cap L^2(\wedge^{p-1} M)$.

Démonstration. First, for each $\phi \in V_k$, we choose η to maximize the quotient $\frac{(\phi,\phi)}{(\eta,\eta)}$.

If $d\eta = \phi$, we use the Hodge decomposition of $\eta = d\alpha + \delta\beta + \gamma$ and choose $\eta_0 = \delta\beta$. Recall that we have $i_\nu \eta_0 = 0$, as said in Theorem 29.

It follows then that

$$inf_{V_k} sup_{V_k - \{0\}} \{ \frac{(\phi, \phi)}{(\eta, \eta)} : d\eta = \phi \} = inf_{V_k} sup_{V_k - \{0\}} \{ \frac{(\phi, \phi)}{(\eta_0, \eta_0)} \},$$

and this is equal to

$$inf_{W_k} sup_{\eta_0 \in W_k - \{0\}} \{ \frac{(d\eta_0, d\eta_0)}{(\eta_0, \eta_0)} \}$$

where W_k ranges over subspaces of dimension k of p-forms satisfying the first boundary condition $i_\nu \eta_0 = 0$.

But this is exactly the variational characterization of the spectrum of coexact (p-1)-forms with absolute boundary condition, which allow to conclude.

□

It is of fundamental importance to note that ϕ or η are *not* suppose to satisfy absolute (or relative) boundary conditions. Theorem 40 has some very useful consequences.

1. If ϕ is an exact p-form (and in the sequel, it will be the restriction to a domain of an exact p-form), then in order to estimate $\lambda'_{1,p}$ we can choose for V the vector space generated by ϕ, and we get

$$\lambda'_{1,p} \leq sup_\eta \{ \frac{(\phi, \phi)}{(\eta, \eta)} : d\eta = \phi \} \tag{10}$$

2. Moreover, the supremum is achieve if η is coexact. It follows that, if η is coexact and $d\eta = \phi$, we get
$$\frac{(\phi, \phi)}{(\eta, \eta)} \geq \lambda'_{1,p} \tag{11}$$
3. If now ϕ_1 is an exact eigenform for $\lambda'_{1,p}$ (with absolute boundary conditions) and if η_1 is coexact, satisfies the absolute boundary conditions and is such that $d\eta_1 = \phi_1$, then
$$\lambda'_{1,p} = \frac{(\phi_1, \phi_1)}{(\eta_1, \eta_1)} \geq \frac{(\phi_1, \phi_1)}{(\eta, \eta)} \tag{12}$$
for each η such that $d\eta = \phi_1$.

Proof of Theorem 39 Let α be an eigenform for $\lambda'_{1,p}(M)$ and β with $d\beta = \alpha$. We have
$$\lambda'_{1,p} \geq \frac{(\alpha, \alpha)}{(\beta, \beta)}.$$

The goal will be to find a "good" β for this equation, where "good" mean that we can find β with (β, β) bounded from above thanks to the informations we have from the U_i and from the U_{ij}.

Let denote by α_i the restriction from α to U_i. It follow from (11) that there exists β_i coexact on U_i with $d\beta_i = \alpha_i$. and
$$\mu(U_i) \leq \frac{(\alpha_i, \alpha_i)}{(\beta_i, \beta_i)} \leq \frac{(\alpha, \alpha)}{(\beta_i, \beta_i)} \tag{13}$$
so that
$$(\beta_i, \beta_i) \leq \frac{1}{\mu(U_i)}(\alpha, \alpha) \tag{14}$$

Now, suppose (what is in general not correct !) that there exist β on M such β_i is the restriction of β to U_i. It would become easy to have a minoration of $\lambda_{1,p}$, because
$$(\beta, \beta) \leq \sum_{i=1}^{k} (\beta_i, \beta_i) \leq (\alpha, \alpha) \sum_{i=1}^{k} \frac{1}{\mu(U_i)},$$
so
$$\lambda'_{1,p}(M) \geq \frac{1}{\sum_{i=1}^{k} \frac{1}{\mu(U_i)}}.$$

As this is not true, the idea is to correct the situation to make this in some sense possible.

The difficulty comes from the fact that, in general, the restriction of β_i and β_j to U_{ij} do not coincide. We will try to replace β_i by $\bar{\beta}_i$ with this property. We are looking at solutions of type

$$\bar{\beta}_i = \beta_i + d\eta_i \tag{15}$$

where η_i is defined over U_i and together with a control form above of $\|d\eta_i\|$.

To do this, we will take advantage of the fact that, on U_{ij}, $d\beta_i = \alpha = d\beta_j$, so that $d(\beta_j - \beta_i) = 0$. We can use the hypothesis that there is no cohomology of degree $(p-1)$ in U_{ij}, which implies the exactness of $\beta_j - \beta_i$. There exists γ_{ij} defined on U_{ij}, with

$$d\gamma_{ij} = \beta_j - \beta_i,$$

and γ_{ij} coexact.

The goal will be to write

$$\gamma_{ij} = \tau_i - \tau_j$$

with τ_i defined on U_i and then to write

$$\bar{\beta}_i = \beta_i + d\tau_i, \ i = 1, ..., k.$$

We introduce a partition of unity $\{\rho_i\}_{i=1}^k$ subordinated to the covering (U_i) and write

$$\tau_i = \sum_{l=1}^{k} \rho_l \gamma_{il}.$$

Observe that τ_i is well defined on U_i by extension by 0 of $\rho_l \gamma_{il}$ on U_i.

First, we have to show that $\bar{\beta}_i$ and $\bar{\beta}_j$ coincide on U_{ij}.

We have

$$\bar{\beta}_i - \bar{\beta}_j = (\beta_i - \beta_j) + d(\tau_i - \tau_j) = -d\gamma_{ij} + d(\tau_i - \tau_j).$$

So, we have to show that $\tau_i - \tau_j = \gamma_{ij}$ on U_{ij}.

Let $x \in U_{ij}$. Because the partition of unity $\{\rho_i\}_{i=1}^k$ is subordinated to the covering (U_i), and because of the hypothesis (no intersection of order 3), only $(\rho_i(x)$ and $\rho_j(x)$ may differ from 0. We get

$$\tau_i(x) - \tau_j(x) = \rho_j(x)\gamma_{ij}(x) - \rho_i(x)\gamma_{ji}(x) = (\rho_i(x) + \rho_j(x))\gamma_{ij}(x) = \gamma_{ij}(x).$$

To have the right estimate, we have to control the norm $\|\bar{\beta}_i\|$. We have

$$\|\beta_i + d\tau_i\|^2 \leq 2(\|\beta_i\|^2 + \|d\tau_i\|^2) \tag{16}$$

and we already know how to control the term $\|\beta_i\|$.

We also have

$$\|d\tau_i\| = \|\sum_{l=1}^{k}(d\rho_l \wedge \gamma_{il} + \rho_l d\gamma_{il})\| \tag{17}$$

This is the reason we need to control the L^∞-norm of the partition of unity. With this a priori estimate, it is enough to control $\|\gamma_{ij}\|$ and $\|d\gamma_{ij}\|$.

As $\|d\gamma_{ij}\| = \|\beta_j - \beta_i\|$, the above mentioned estimate of $\|\beta_i\|$ allows to estimate $\|d\gamma_{ij}\|$.

To estimate $\|\gamma_{ij}\|$, we use again (11). As γ_{ij} is coexact, we have

$$\|\gamma_{ij}\|^2 \leq \frac{1}{\mu(U_{ij})}\|d\gamma_{ij}\|^2 \tag{18}$$

which allows to get the desired estimate of $\lambda'_{1,p}(M)$.

4 Lecture 4 : Small eigenvalues under collapsing

In this lecture, we will investigate the question :

"Does a family of compact Riemannian manifolds with bounded diameter and curvature and volume (or injectivity radius) converging to 0 has nonzero eigenvalues for p-forms converging to 0 ?"

This question would justify a series of lectures for itself, because of a lot a recent and interesting developments (see [CC1], [CC2], [Ja1], [Ja2],[Ja3], [Lo1], [Lo2], [Lo3]). The goal of the section is just to give an overview of the problem. Note that a related question, we will not study at all, is about the adiabatic limits (see [Fo]).

Recall that Example 23 shows that we may have small eigenvalue as the injectivity radius (or the volume) goes to 0, but that the example of a product $M \times S^1$ shows that it may have no small eigenvalue.

4.1 A few words about collapsing

We begin by some geometrical considerations :

Definition 41. *We say that a compact manifold M admits a collapsing if there exists two positive constant a and d and a family $\{g_i\}_{i=1}^{\infty}$ of Riemannian metrics such that $|K(g_i)| \leq a$, $diam(g_i) \leq d$ and $inj(g_i) \to 0$ as $i \to \infty$ (or equivalently, $vol(g_i) \to 0$ as $i \to \infty$).*

Notation In the sequel, we will call *a* (a, d)*-metric* a Riemannian metric with sectional curvature $|K| \leq a$ and diameter $diam \leq d$.

It is in general not easy to decide if a manifold admit or not a collapsing, but this depends clearly of its topology. As example, if a manifold admits a collapsing, its minimal volume is 0 and all its characteristic numbers have to vanish. The metric description of a collapsing manifold was investigate of lot during the 80', and a quit complete, but very complicated, description is given in the paper of Cheeger-Fukaya-Gromov [CFG]. However, for the purpose of these notes, it is easier to give a partial result of Fukaya [Fu1]

Theorem 42. *Let* (M_i, g_i) *a sequence of compact Riemannian manifolds of dimension* n *with sectional curvature and diameter uniformly bounded, and* (N, h) *a compact manifold of dimension* $m < n$*. If* (M_i, g_i) *converge to* (N, h) *for the Gromov-Hausdorff distance, then, for* i *large enough, there is a fiber bundle structure* $\pi : M_i \to N$ *whose fiber is an infranilmanifold.*

The prototype of an infranilmanifold is a torus (or a quotient of a torus) and the prototype of such a bundle is a torus bundle ; (note however that not all torus bundle may collapse). A simple example of collapsing manifold is the product of a fixed Riemannian manifold N with a flat torus whose injectivity radius goes to 0 (which does not affect its curvature !), and roughly speaking, we may think to a collapsing manifold as a manifold which is *locally* a product of a manifold with a flat torus (or a nilmanifold). The simplest non trivial examples we can imagine are the following :

- At first, a S^1-bundle on a manifold N ;
- Then a torus bundle on a simple manifold N, as example a circle.

And these examples were the first one's to be deeply investigated from the point of view of the spectral theory ([CC1], [CC2] for the S^1-bundle and [Ja1] for torus bundle over a circle).

The easiest result to present is the case of S^1-bundle, and it allows to ask a lot of open questions.

4.2 The case of S^1-bundle

The goal is to give the main steps of the paper [CC2]. We will state a precise result at the end of the first step.

Step 1 There is an easy but very useful fact we can use when we study the spectrum from a qualitative viewpoint (and to decide if there are small eigenvalues and even to have informations about their asymptotic behaviour is a prototype of qualitative question). It is the property that two quasi-isometric Riemannian manifolds have comparable spectrum. This is a result du to J. Dodziuk (see [Do])

Definition 43. *Let M be a manifold and $\tau > 1$. Two Riemannian metrics g_1 and g_2 on M are said τ-quasi-isometric if, for each p in M and $v \in T_pM$, then*

$$\frac{1}{\tau} \leq \frac{g_1(p)(v,v)}{g_2(p)(v,v)} \leq \tau.$$

Theorem 44. *Let M be a compact manifold of n dimension, and g_1, g_2 two Riemannian metric τ-quasi-isometric on it. Then, for each $0 \leq p \leq n$ and $k > 0$, we have*

$$\frac{1}{\tau^{n+2p+1}}\lambda_{k,p}(M,g_1) \leq \lambda_{k,p}(M,g_2) \leq \tau^{n+2p+1}\lambda_{k,p}(M,g_2) \tag{19}$$

If $S^1 \hookrightarrow (M,g') \to (N,h')$ is a S^1 principal bundle with a (a',d')-metric on M, we show in the first step that we can replace this by $S^1 \hookrightarrow (M,g) \to (N,h)$, where g is a (a,d)-metric (a,d depending only on a',d') and g,h are τ-quasi-isometric to g' and h' respectively, τ depending only on a',d', such that the bundle

$$S^1 \hookrightarrow (M,g) \xrightarrow{\pi} (N,h)$$

has very nice properties :

1. The principal bundle $S^1 \hookrightarrow (M,g) \xrightarrow{\pi} (N,h)$ is a Riemannian submersion ;
2. The fibers are geodesics all of the same length ϵ ;
3. The action of S^1 is isometric ;
4. If ω is the vertical 1-form of norm 1 associated to the action of S^1, then $d\omega = \epsilon\pi^*(e(M))$, where $e(M)$ is the harmonic representant of the Euler class of the bundle.

Definition 45. *Such a couple (g,h) of Riemannian metric will be called a (a,d)-adapted metric.*

Another way to understand the first step is that we show that the situation we want to study is very similar to the Example 23. Here, the Euler class of the Hopf-bundle was the Kähler form of the complex projective space.

In the context of (a,d)-adapted metric, we can state our result, which will be also true in general because of the quasi-isometry property. In the sequel, we set $m_p = b_p(N) + b_{p-1}(N) - b_p(M)$, which measure in some sense the defect of M to be a product.

Theorem 46. *If ϵ is small enough, there exists $C_i = C_i(n,a,d) > 0$, $i = 1,2$, such that*

$$0 < \lambda_{k,p}(M,g) \leq C_1(\epsilon\|e\|_2)^2,\ 1 \leq k \leq m_p; \tag{20}$$

$$\lambda_{m_p+1,p}(M,g) \geq C_2. \tag{21}$$

Theorem 47. *If ϵ is small enough, there exists $C_i = C_i(n, a, d) > 0$, $i = 1, 2$, such that, if $e(M) \neq 0$, then*

$$C_1(\epsilon\|e\|_2)^2 < \lambda_{1,1}(M, g) \leq C_2(\epsilon\|e\|_2)^2. \tag{22}$$

Theorem 48. *If ϵ is small enough, there exists $C_i = C_i(n, a, d) > 0$, $i = 1, 2$, such that, if $e(M) \neq 0$ and if $dim H^2(N, \mathbb{R}) = 1$, then*

$$C_1(\epsilon\|e\|_2)^2 < \lambda_{k,p}(M, g) \leq C_2(\epsilon\|e\|_2)^2, \ 1 \leq k \leq m_p. \tag{23}$$

Some comments

1. The first clear point is that it cannot be more small eigenvalues for p-forms as the maximal number of p-harmonic forms (the case of the product), and that the number of small nonzero p-eigenvalues measure precisely the defect to be product ;
2. A second question is to estimate asymptotically the small eigenvalues. It is clear that it is not enough to consider ϵ (which correspond to the injectivity radius), but that we have to take account of $\|e\|^2$ (see Example 49 to understand why) ;
3. Then, Theorem 48 say that, if the 2 nd cohomology of the basis is not complicated, it is indeed true that $(\epsilon\|e\|)^2$ give a good estimate of the small eigenvalues, Theorem 47 says that this is always true for $\lambda_{1,1}$, and Theorem 46 says that in general it gives only an upper bound. An (non easy) example given in [CC2] explains why we cannot hope to get a lower bound.

Example 49. *There is an isometric action of the finite group $\mathbb{Z}/q\mathbb{Z}$ on S^{2n+1} given by Example 23 if we see $\mathbb{Z}/q\mathbb{Z}$ as the q-root of unity.*

Let denote by L_q the quotient of S^{2n+1} by this action, with the induced metric. We have a S^1-bundle

$$S^1 \hookrightarrow L_q \to \mathbb{C}P^n$$

with totally geodesic fiber and injectivity radius π/q going to 0 as $q \to \infty$. However, there are no small eigenvalues on L_q, because each eigenform may be lift to an eigenform of the round sphere. This come from the fact that the Euler class e_q of L_q has a norm going to ∞ as $q \to \infty$, so that the product $(inj(L_q)\|e_q\|_2)$ stay bounded away from 0.

Step 2 In the context of an (a, d)-adapted metric, this step is very natural : it consists to show that only the S^1-invariant p-forms may produce small eigenvalues. This is well explain in the paper of P. Jammes [Ja2].

Step 3 In this step, we have to prove Theorem 46, 47 and 48 for S^1-invariant forms. We just explain here the beginning of the calculations.

Let ψ be a S^1-invariant p-form on M. We can write

$$\psi = \pi^*\alpha + \pi^* a \wedge \omega \tag{24}$$

where we recall that ω is the vertical 1-form of norm 1 associated to the action of S^1. Since we have $d\omega = \epsilon e$, a standard calculation shows that

$$\|\psi\|_2^2 = \epsilon(\|\alpha\|_2^2 + \|a\|_2^2) \tag{25}$$

$$\|d\psi\|_2^2 = \epsilon(\|d\alpha + (-1)^{p-1}\epsilon(a \wedge e)\|_2^2 + \|da\|_2^2) \tag{26}$$

$$\|\delta\psi\|_2^2 = \epsilon(\|\delta\alpha\|_2^2 + \|\delta a + (-1)^{np}\epsilon * (*\alpha \wedge e)\|_2^2) \tag{27}$$

Now the goal is to control the apparition of small eigenvalues as $\epsilon \to 0$. It is intuitively clear that and it is not difficult to show that we have only to look at the lift of harmonic forms of the basis manifold N.

So we lift $b_p(N)+b_{p-1}$ harmonic forms of N to M. But $b_p(M)$ of them will give harmonic forms. We will see that the other ones gives m_p small eigenvalues.

If α and a are harmonic forms, we get the expression

$$\|\psi\|_2^2 = \epsilon(\|\alpha\|_2^2 + \|a\|_2^2) \tag{28}$$

$$\|d\psi\|_2^2 = \epsilon^3\|a \wedge e\|_2^2 \tag{29}$$

$$\|\delta\psi\|_2^2 = \epsilon^3\| * \alpha \wedge e\|_2^2 \tag{30}$$

If we look at $R(\psi)$, we note that as a and α are harmonic of L^2-norm ≤ 1, their L^∞ norm is controled by the geometry of the manifold, so that we get

$$R(\psi) \leq C\epsilon^2\|e\|_2^2. \tag{31}$$

It is much more technical to understand to what extend this gives the good asymptotic, and this is the purpose of the paper [CC2] to do it.

4.3 Other developments from J. Lott and P. Jammes

The work of P. Jammes In a series of very interesting papers ([Ja1], [Ja2], [Ja3]), Pierre Jammes has investigated the following question : there is only one way to collapse a circle, but as soon as we collapse a bundle whose fiber has dimension > 1, there are a lot of different way to collapse. Does the way of collapsing affect the apparition of small eigenvalues ?

P. Jammes investigated mainly the torus bundles, and the answer is clear : the way of collapsing affects a lot the apparition of small eigenvalues. The things are so tricky that the works of P. Jammes leads to a lot of new open questions. One can find a small survey in [Ja4]. Let us just mention two in my opinion very spectacular results of P. Jammes (see [Ja2])

Theorem 50. *Let $k \geq 1$. For each Riemannian manifold (N, h) such that $b_2(N) \geq k$, there exists a principal T^k- torus bundle on N, a family of Riemannian metric $(g_\epsilon)_{0<\epsilon<1}$ on M, and two positive constants $C = C(k, h)$, $\epsilon_0 = \epsilon_0(k, h)$ such that :*

$$|K(M, g_\epsilon)| \leq a; diam(M, g_\epsilon) \leq d \text{ and } vol(M, g_\epsilon) = \epsilon \tag{32}$$

and, for $p = 1, 2$ and ϵ small enough,

$$\lambda_{1,p}(M, g_\epsilon) \leq C(inj(M, g_\epsilon)^{2k}. \tag{33}$$

This result show that (at least for the small degrees) the dependance of λ_1, p with the injectivity depends on the dimension, what was totally unclear when we study the S^1-bundle. This lead also to the question (see [Ja4])

Open question 4. *For a compact Riemannian manifold M of dimension n with bounded sectional curvature by k and diameter bounded from above by d, do we have an estimation of the type*

$$\lambda_{1,p} \geq C(n, d, k)(inj(M))^{an+b}$$

with a, b universal constants.

Theorem 50 leads to try to estimate the spectrum with respect to the volume. P. Jammes has also a result in this direction

Theorem 51. *Let $T^k \hookrightarrow (M, g) \xrightarrow{\pi} (N, h)$ a T^k-principal bundle so that $|K(g)| \leq a$ and $diam(g) \geq d$. If (M, g) is ϵ-Hausdorff close to (N, h), then and ϵ small enough, then*

$$\lambda_{1,1}(M, g) \geq C(n, a, d)Vol^2(M, g).$$

This lead to the two natural questions (see [Ja4])

Open question 5. *In the context of Theorem 51, what about the other degrees ?*

Open question 6. *In general (diameter and sectional curvature bounded), could we hope to have a minoration of $\lambda_{1,p}$ with respect to Vol^2 ?*

Two last general questions in these directions are the following

Open question 7. *Is it possible to generalize Theorem 25 (Chanillo-Trèves) with an hypothesis on the Ricci curvature and not on the sectional one's ?*

Open question 8. *Does each compact manifold of dimension ≥ 3 admit a family of metric with $Ric(g)diam(g)^2 \geq C$ and arbitrarily small $\lambda_{1,p}$. Recall that the hypothesis $Ric(g)diam(g)^2 \geq C$ implies that $\lambda_{1,0}$ is bounded away from 0 (Theorem 12).*

The work of J. Lott The approach of J. Lott ([Lo1], [Lo2]) is a generalization to the p-forms spectrum of the work of Fukaya for functions [Fu2]. If a family of Riemannian manifolds (M_i, g_i) collapses to a limit X, is it possible to construct a limit operator on X so that the spectrum of X gives the limit of the spectrum of (M_i, g_i) ? In fact, in the case of form, the idea is to construct a vector bundle on the limit, and then an operator on this limit.

The results of Lott are related to the question of the apparition of small eigenvalues as follow. The number of small eigenvalues for p-forms is the small eigenvalues which are not 0. But all the small eigenvalues for p-forms (which are 0 or not) converge to 0 at the limit and give a contribution to the kernel of the limit operator. So, if we are able to investigate the dimension of the kernel of this operator, we get a qualitative information about the number of small eigenvalues, and this is well illustrated in Lott's papers. This is also quit difficult in general.

Open question 9. *The paper [Lo2] is not published, perhaps because it contains some gaps in some proofs, but it is in any case very interesting. It would be interesting to study this paper and to illustrate it with examples.*

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