Let M be a compact manifold without boundary.
To every Riemannian metric g on M, we associate its Laplace-Beltrami operator Δ_g and denote by $\lambda_1(M,g)$ the smallest positive eigenvalue of Δ_g :

$$\lambda_1(M,g) = \inf_{\int_M f v_g = 0} \frac{\int_M |\nabla f|^2 v_g}{\int_M f^2 v_g}.$$

Problem : To optimize the functional $g \mapsto \lambda_1(M,g)$.

Since $\lambda_1(M,kg) = \frac{1}{k}\lambda_1(M,g)$, a normalization is needed. We restrict the functional to

$$\mathcal{R}(M) = \{\text{metrics of volume 1 on } M\}.$$

Proposition : $\inf_{g\in\mathcal{R}(M)} \lambda_1(M,g) = 0$.

Hersch (1970) : $\forall g \in \mathcal{R}(\mathbb{S}^2)$,

$$\lambda_1(\mathbb{S}^2, g) \leq 8\pi,$$

where the equality holds iff $g \cong can$.

Yang - Yau (1980) : If M is a compact orientable surface, then

$$\sup_{g \in \mathcal{R}(M)} \lambda_1(M, g) \leq 8\pi(\text{genus}(M) + 1).$$

E.-Ilias (1984) :

$$\sup_{g \in \mathcal{R}(M)} \lambda_1(M, g) \leq 8\pi \left[\frac{\text{genus}(M) + 3}{2}\right].$$

Li - Yau (1982) : 1) $\forall g \in \mathcal{R}(\mathbb{R}P^2)$,

$$\lambda_1(\mathbb{R}P^2, g) \leq 12\pi$$

where the equality holds iff $g \cong can$.
2) If M is a compact nonorientable surface, then

$$\sup_{g \in \mathcal{R}(M)} \lambda_1(M, g) \leq 24\pi(\text{genus}(M) + 1).$$

Colbois-Dodziuk (1994) : If $\dim M \geq 3$, then

$$\sup_{g \in \mathcal{R}(M)} \lambda_1(M, g) = +\infty.$$

We obtain a relevant topological invariant of surfaces by setting

$$\Lambda(M) = \sup_{g \in \mathcal{R}(M)} \lambda_1(M, g) = \sup_g \lambda_1(M, g) A(M, g)$$

where $A(M, g)$ is the area of (M, g).

Natural questions :

1. How does $\Lambda(M)$ behave in terms of the genus of M ?
2. Can one determine $\Lambda(M)$?
3. Does the supremum $\Lambda(M)$ achieved ? what can one say about the (eventual) extremal metrics ?

Concerning question 1 :

Colbois-E.(2003) : $\Lambda(M)$ is an increasing function of the genus of M with a linear growth rate.

Concerning questions 2 and 3 :

The results of Hersch and Li - Yau read :

$$\Lambda(\mathbb{S}^2) = \lambda_1(\mathbb{S}^2, can)A(\mathbb{S}^2, can) = 8\pi$$

and

$$\Lambda(\mathbb{R}P^2) = \lambda_1(\mathbb{R}P^2, can)A(\mathbb{R}P^2, can) = 12\pi.$$

Moreover, in each case, can is the "'only"' extremal metric.

More about question 3 :

Nadirashvili (1996) : if M has genus 1 (i.e torus $\mathbb{T}^2$ or Klein bottle $\mathbb{K}^2$), then the supremum $\Lambda(M)$ is achieved by at least one regular metric.

Extremal metrics

Despite the non-differentiability of the functional $g \mapsto \lambda_1(M,g)$ with respect to metric deformations, a natural notion of critical metric can be introduced. Indeed, perturbation theory enables us to prove that, for any analytic deformation g_ε of a metric g, the function $\varepsilon \mapsto \lambda_1(M,g_\varepsilon)$ always admits left and right derivatives at $\varepsilon = 0$ such that

$$\frac{d}{d\varepsilon}\lambda_1(M,g_\varepsilon)\Big|_{\varepsilon=0^+} \le \frac{d}{d\varepsilon}\lambda_1(M,g_\varepsilon)\Big|_{\varepsilon=0^-}.$$

The metric g is then said to be *critical* for the functional λ_1 if, for any analytic volume preserving invariant deformation g_ε of g, one has

$$\frac{d}{d\varepsilon}\lambda_1(M,g_\varepsilon)\Big|_{\varepsilon=0^+} \le 0 \le \frac{d}{d\varepsilon}\lambda_1(M,g_\varepsilon)\Big|_{\varepsilon=0^-},$$

which means that

$$\lambda_1(M,g_\varepsilon) \le \lambda_1(M,g) + o(\varepsilon) \quad \text{as } \varepsilon \to 0.$$

In particular, if a metric g is a local minimizer or a local maximizer of λ_1 on $\mathcal{R}(M)$, then g is a critical metric in the sense of this definition.

E.-Ilias (2000) : A metric g is critical for λ_1 iff there exists a finite family $h_1, \cdots, h_d$ of first eigenfunctions of Δ_g satisfying

$$\sum_{i \leq d} dh_i \otimes dh_i = g. \tag{1}$$

Condition (1) means that the map

$$h = (h_1, \cdots, h_d) : (M, g) \to \mathbb{R}^d$$

is a an isometric immersion.

Takahashi (1966) : Let $h = (h_1, \cdots, h_d) : (M, g) \to \mathbb{R}^d$ be an isometric immersion. The following conditions are equivalent

1. $\exists \lambda\,;\ \Delta_g h_i = \lambda h_i$
2. $h(M)$ is a minimal submanifold of the sphere $\mathbb{S}^{d-1}\left(\sqrt{\frac{2}{\lambda(M,g)}}\right)$. In particular,

$$\sum_{i \leq d} h_i^2 = \frac{2}{\lambda(M, g)}. \tag{2}$$

Hence, g is critical iff (M, g) can be realized as a minimal submanifold of a sphere by the means of its first eigenfunctions.

E.-Ilias (1986) : If (M, g) is isometrically immersed as a minimal submanifold of a sphere by the first eigenfunctions (or, equivalently, if g is a critical metric of λ_1 in $\mathcal{R}(M)$), then

i) g is "unique" in its conformal class,
ii) g maximizes λ_1 over its conformal class, under the "volume preserving" constraint.
iii) For any metric g' conformal to g one has

$$\mathrm{Isom}(M, g') \subset \mathrm{Isom}(M, g).$$

Consequence : On $\mathbb{S}^2$ (resp. $\mathbb{R}P^2$) the standard metric can is, up to dilatations, the unique critical metric.

What about the Torus ?

E.-Ilias (2000) : There exist (up to isometries) exactly two possible immersions of a torus as a minimal surface of a sphere by the first eigenfunctions :

- Clifford torus : $h_{cl} : \mathbb{R}^2/\mathbb{Z}^2 \hookrightarrow \mathbb{S}^3$, with

$$h_{cl}(x,y) = \frac{1}{\sqrt{2}}(e^{2i\pi x}, e^{2i\pi y}),$$

- Equilateral torus : $h_{eq} : \mathbb{R}^2/\mathbb{Z}(1,0)\oplus\mathbb{Z}(\frac{1}{2},\frac{\sqrt{3}}{2}) \hookrightarrow \mathbb{S}^5$, with

$$h_{eq}(x,y) = \frac{1}{\sqrt{3}}(e^{4i\pi y/\sqrt{3}}, e^{2i\pi(x-y/\sqrt{3})}, e^{2i\pi(x+y/\sqrt{3})}).$$

Consequently, the corresponding flat metrics g_{cl} et g_{eq} are, up to dilatations, the only critical metrics of λ_1 on the torus $\mathbb{T}^2$.

Consequence :

$$\Lambda(\mathbb{T}^2) = \lambda_1(\mathbb{T}^2, g_{eq})A(\mathbb{T}^2, g_{eq}) = \frac{8\pi^2}{\sqrt{3}},$$

and g_{eq} is the only maximizer.

What about the Klein bottle $\mathbf{K}$?

To every $a > 0$, we associate the rectangular lattice $\Gamma_a = \mathbb{Z}(2\pi, 0) \oplus \mathbb{Z}(0, a) \subset \mathbb{R}^2$ and denote by $\tilde{g}_a$ the induced flat metric on the torus $\mathbb{T}^2_a \simeq \mathbb{R}^2/\Gamma_a$. The Klein bottle $\mathbf{K}$ is diffeomorphic to the quotient of $\mathbb{T}^2_a$ by the involution $s : (x, y) \mapsto (x + \pi, -y)$. We denote by g_a the flat metric induced on $\mathbf{K}$ from the covering $(\mathbb{T}^2_a, \tilde{g}_a) \to \mathbf{K}$.
Any Riemannian metric on $\mathbf{K}$ is conformal to one of the metrics g_a.

If $g = f g_a$ is a critical metric on $\mathbf{K}$, then

$$\mathrm{Isom}(\mathbf{K}, g_a) \subset \mathrm{Isom}(\mathbf{K}, g).$$

The group $\mathrm{Isom}(\mathbf{K}, g_a)$ contains the $\mathbb{S}^1$-action

$$(x, y) \mapsto (x + t, y), \qquad t \in [0, \pi].$$

Therefore, f does not depend on the variable x.

Consider a metric $g = f(y)g_a$. The Laplacian Δ_g can be identified with the operator $-\frac{1}{f(y)}\left(\partial_x^2 + \partial_y^2\right)$ acting on Γ_a-periodic and s-invariant functions of $\mathbb{R}^2$. Separating variables, the eigenspaces are generated by functions of the form $\varphi_k(y)\cos kx$ and $\varphi_k(y)\sin kx$ where, $\forall k$, φ_k is periodic of period a, $\varphi_k(-y) = (-1)^k\varphi_k(y)$ et

$$\varphi_k'' = (k^2 - \lambda f)\varphi_k$$

for some $\lambda > 0$.

Since a first eigenfunction has exactly two nodal domains (Courant), the first eigenspace is generated by :
$\varphi_0(y)$, $\varphi_1(y)\cos x$, $\varphi_1(y)\sin x$,
$\varphi_2(y)\cos 2x$, $\varphi_2(y)\sin 2x$.
with, unless they are identically zero,
- φ_0 and φ_1 admit exactly two zeros in $[0, a)$,
- φ_2 does not vanish in $[0, a)$.

Now, we look for an isometric immersion $h = (h_1, \cdots, h_d) : (\mathbf{K}, f(y)g_a) \to \mathbb{S}^{d-1}(r)$ by the first eigenfunctions. Without lack of generality, one may assume that $r = 1$ and that $h_1, \cdots, h_d$ are linearly independent, hence $d = 4$ ou 5.

$d = 4$: There exists $\rho \in O(4)$ s. t.

$$\rho \circ h = (\varphi_1(y)e^{ix}, \varphi_2(y)e^{2ix}),$$

with

$$\varphi_1^2 + \varphi_2^2 = 1$$

and, since h is isometric, $|\partial_y h|^2 = |\partial_x h|^2 = f$, which means

$$\varphi_1'^2 + \varphi_2'^2 = \varphi_1^2 + 4\varphi_2^2 = f.$$

But this contradicts $\varphi_1^2 + \varphi_2^2 = 1$ which tells us that φ_1 achieves its max at the same point(s) where φ_2 achieves its min, and reciprocally.

Therefore, $d = 5$ and there exists $\rho \in O(5)$ s.t.

$$\rho \circ h = (\varphi_0(y), \varphi_1(y)e^{ix}, \varphi_2(y)e^{2ix})$$

with

$$\varphi_0^2 + \varphi_1^2 + \varphi_2^2 = 1$$

and

$$\varphi'^2_0 + \varphi'^2_1 + \varphi'^2_2 = \varphi_1^2 + 4\varphi_2^2 = f.$$

In particular, φ_1 and φ_2 are solutions of the system

$$\begin{cases} \varphi_1'' = (1 - 2\varphi_1^2 - 8\varphi_2^2)\varphi_1, \\ \\ \varphi_2'' = (4 - 2\varphi_1^2 - 8\varphi_2^2)\varphi_2; \end{cases}$$

Proposition Let $a > 0$ and f a positive periodic function of period a. the following are equivalent :

(I) The metric $g = f(y)g_a$ on $\mathbf{K}$ is critical for λ_1 on $\mathcal{R}(\mathbf{K})$.

(II) There exists a homothetic minimal immersion $h : (\mathbf{K}, \mathbf{g}) \to \mathbb{S}^{\mathbf{d-1}}$ by the first eigenfunctions of Δ_g.

(III) f is proportional to $\varphi_1^2 + 4\varphi_2^2$, where φ_1 and φ_2 are two periodic functions of period a such that :

(a)

$$\begin{cases} \varphi_1'' = (1 - 2\varphi_1^2 - 8\varphi_2^2)\varphi_1 \\ \\ \varphi_2'' = (4 - 2\varphi_1^2 - 8\varphi_2^2)\varphi_2; \end{cases}$$

(b) φ_1 is odd, φ_2 is even and $\varphi_1'(0) = 2\varphi_2(0)$;

(c) φ_1 has exactly two zeros in a period and φ_2 is positive everywhere ;

(d) $\varphi_1^2 + \varphi_2^2 \leq 1$ where the equality holds at exactly two points in a period.

The problem reduces to the study of the system

$$\begin{cases} \varphi_1'' = (1 - 2\varphi_1^2 - 8\varphi_2^2)\varphi_1, \\ \varphi_2'' = (4 - 2\varphi_1^2 - 8\varphi_2^2)\varphi_2, \end{cases} \tag{3}$$

with the initial conditions (Condition (b))

$$\begin{cases} \varphi_1(0) = 0, \quad \varphi_2(0) = p, \\ \varphi_1'(0) = 2p, \, \varphi_2'(0) = 0, \end{cases} \tag{4}$$

où $p \in (0, 1]$ (Condition (d)).

One has to determine the value(s) of p for which the system admits a **periodic** solution (φ_1, φ_2) satisfying (Condition (c)) :

$$\begin{cases} \varphi_1 \text{ has two zeros in a period,} \\ \varphi_2 \text{ is positive everywhere.} \end{cases} \tag{5}$$

Jakobson, Nadirashvili et Polterovich (2003) : For $p = \sqrt{3/8}$, the solution of (3)-(4) is periodic and satisfies (5). The corresponding critical metric g_0 of λ_1 on $\mathcal{R}(\mathbf{K})$ is given by :

$$g_0 = \frac{9 + (1 + 8\cos^2 v)^2}{1 + 8\cos^2 v} \left(du^2 + \frac{dv^2}{1 + 8\cos^2 v} \right)$$

$0 \leq u < \frac{\pi}{2}$, $0 \leq v < \pi$.

El Soufi-Giacomini-Jazar (2005) : Let (φ_1, φ_2) the solution of (3)-(4).

1. $\forall p \in (0,1]$, $p \neq \sqrt{3}/2$, (φ_1, φ_2) is either périodic or quasi-periodic.
2. For $p = \frac{\sqrt{3}}{2}$, (φ_1, φ_2) tends to the origin as $y \to \infty$ (hence, it is neither periodic nor quasi-periodic).
3. For all $p \in (\sqrt{3}/2, 1]$, φ_2 vanishes at least once in each period. Hence, Condition (5) is not satisfied.
4. There exists a countable dense subset $\mathcal{P} \subset (0, \sqrt{3}/2)$, with $\sqrt{3/8} \in \mathcal{P}$, such that the solution (φ_1, φ_2) corresponding to $p \in (0, \sqrt{3}/2)$ is periodic if and only if $p \in \mathcal{P}$.
5. For $p = \sqrt{3/8}$, (φ_1, φ_2) satisfies (5) and, for any $p \in \mathcal{P}$, $p \neq \sqrt{3/8}$, φ_1 admits at least 6 zeros in a period.

Conclusion : The initial value $p = \sqrt{3/8}$ is the only one to correspond to a periodic solution satisfying (5). Therefore, the metric g_0 is, up to dilatations, the unique critical metric of λ_1 on $\mathbf{K}$.

With the existence result of Nadirashvili, we deduce the following

Corollary

$$\Lambda(\mathbf{K}) = \lambda_1(\mathbf{K}, g_0)A(\mathbf{K}, g_0) = 12\pi E(\frac{2\sqrt{2}}{3}),$$

where E is the complete elliptic integral of the second kind ($\Lambda(\mathbf{K}) \approx 13,365\pi$).
Moreover, the metric g_0 is, up to dilatations, the only maximizing metric.

The immersion of the klein bottle $(\mathbf{K}, g_0)$ as a minimal surface of $\mathbb{S}^4$ coincide with the bipolar surface of the Lawson's minimal torus $\tau_{3,1}$ defined in $\mathbb{S}^3$ by

$$(u, v) \mapsto (\cos v \exp(3iu), \sin v \exp(iu)).$$