On the Graphs Whose Cycles of Length Divisible by a Given Number

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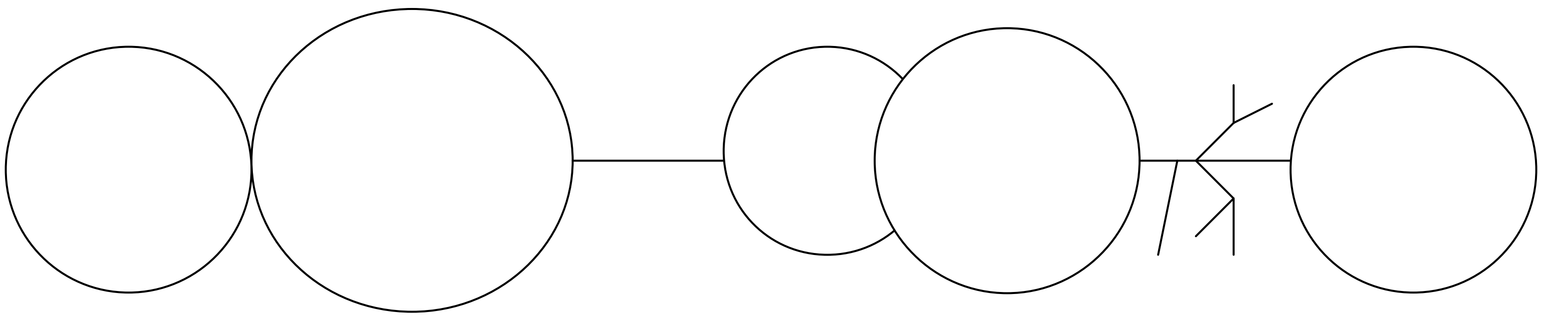
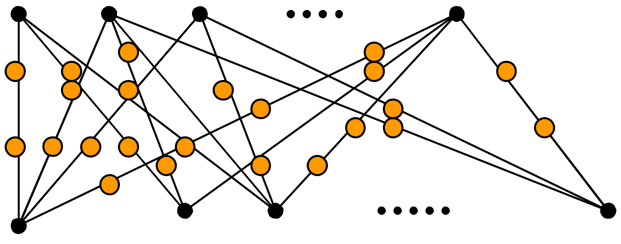
Introduction

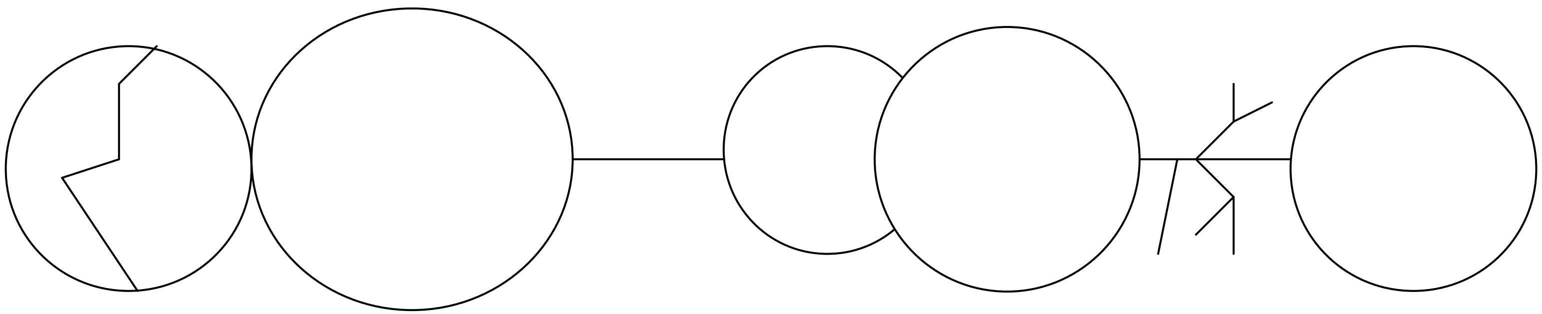
A graph G is said to be a (0 mod l)-cycle graph, if every cycle in G has length divisible by l.





Introduction

For an integer l with $l \geq 2$, ***a graph G is said to be a (0 mod l)-cycle graph, if every cycle in G has length divisible by l.***



Introduction

A trail $P = v_0 v_1 \ldots v_s$ ***in a graph G is called a branch if*** $d_G(v_i) = 2$ ***for*** $1 \le i \le s-1,\ d_G(v_0) \ne 2,\ d_G(v_s) \ne 2$. ***If*** $v_0 = v_s$, ***P is said to be closed. Otherwise, it is said to be open.***

Lemma. ***Let l be an integer with*** $l \geq 3$ ***and G be a connected (0 mod l)-cycle graph with*** $\delta(G) \geq 2$ ***and*** $\Delta(G) \geq 3$, ***then:***

- ***If G has a closed branch C, then*** $l(C) \geq l$.
- ***If G has no closed branch, then G has two branches of length at least*** $\frac{l}{2}$ ***if l is even and at least l if l is odd, which share at least one endvertex.***
- ***If*** $l \neq 4$, ***then G has a pair of adjacent vertices of degree 2.***
- ***If*** $l \notin \{3,4,6\}$, ***then G has three consecutive vertices of degree two.***

Vertex Coloring of Graphs

Let G be a graph. A vertex coloring of G is a function $c: V(G) \longrightarrow L$***, where L is a set with this property: if*** $u, v \in V(G)$ ***are adjacent, then c(u) and c(v) are different.***

A vertex k-coloring is a proper vertex coloring with |L|=k.

The smallest integer k such that G has a vertex k-coloring is called the chromatic number of G and denoted by $\chi(G)$***.***

The List Coloring of Graphs

Let G be a graph and for every $v \in V(G)$, let L(v) denote a list of colors available at v. A **list coloring** or **choice function** is a proper coloring *f* such that for every $v \in V(G)$, $f(v) \in L(v)$.

A graph G is **k-choosable** if every assignment of k-elements lists to the vertices permits a proper list coloring.

The **list chromatic number**, **choice number**, or **choosability of a graph G**, $\chi_l(G)$, is the minimum number k such that G is **k-choosable**.

The List Coloring of Graphs

Theorem.* *A path and cycle are 2-choosable, while an odd cycle is 3-choosable.

Edge Coloring of Graphs

Let G be a graph. An edge coloring of G is a function $c: E(G) \longrightarrow L$, where L is a set with this property: if $s, t \in E(G)$ are adjacent, then c(t) and c(s) are different.

An edge k-coloring is an edge coloring with |L|=k.

The smallest integer k such that G has an edge k-coloring is called the edge-chromatic number of G and denoted by $\chi'(G)$.

Edge Coloring of Graphs

Vizing's Theorem.

If G is a graph, then $\Delta(G) \leq \chi'(G) \leq \Delta(G)+1$**.**

The List Edge-Coloring of Graphs

Let G be a graph and for every $e \in E(G)$, let L(e) denote a list of colors available for e. A list edge-coloring is a proper edge-coloring f with f(e) chosen from L(e) for each e.

The edge-choosablity, $\chi_l^{'}(G)$, is the minimum k such that every assignment of lists of size k yields a proper list edge-coloring.

- ***For every graph G,*** $\Delta(G) \le \chi_l^{'}(G)$.

The List Edge-Coloring of Graphs

Theorem.

A path and even cycle are 2-edge-choosable, while an odd cycle is 3-edge-choosable.

Total Coloring of Graphs

Let G be a graph. A total coloring of G is a function $c: V(G) \cup E(G) \longrightarrow L,$ ***where L is a set with this property that color objects have different colors when they are adjacent or incident.***

A total k-coloring is a total coloring with |L|=k.

The smallest integer k such that G has a total k-coloring is called the total-chromatic number of G and denoted by $\chi''(G)$.

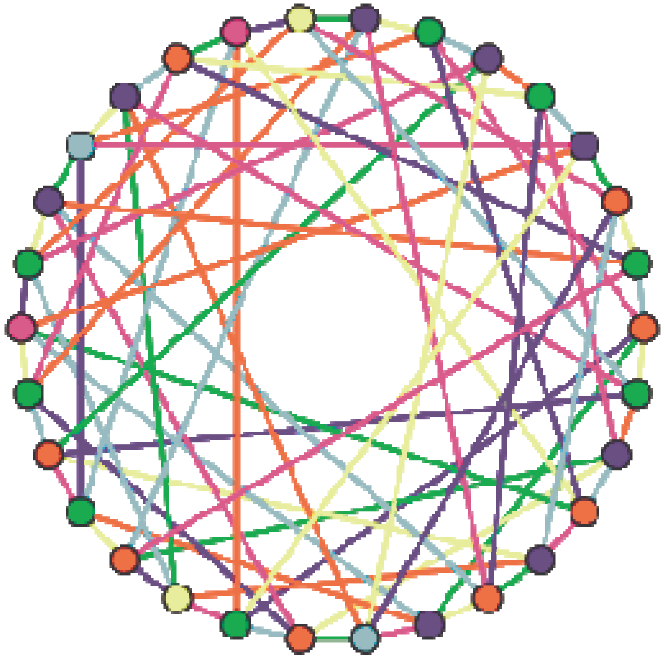
- ***For every graph G,*** $\chi''(G) \geq \Delta(G) + 1$.

Total Coloring Conjecture.

For every graph G, $\chi''(G) \le \Delta(G) + 2.$

With a prize 10,000,000 rials.

Total Coloring of Graphs



$$\chi''(G) = \Delta(G) + 1 = 6$$

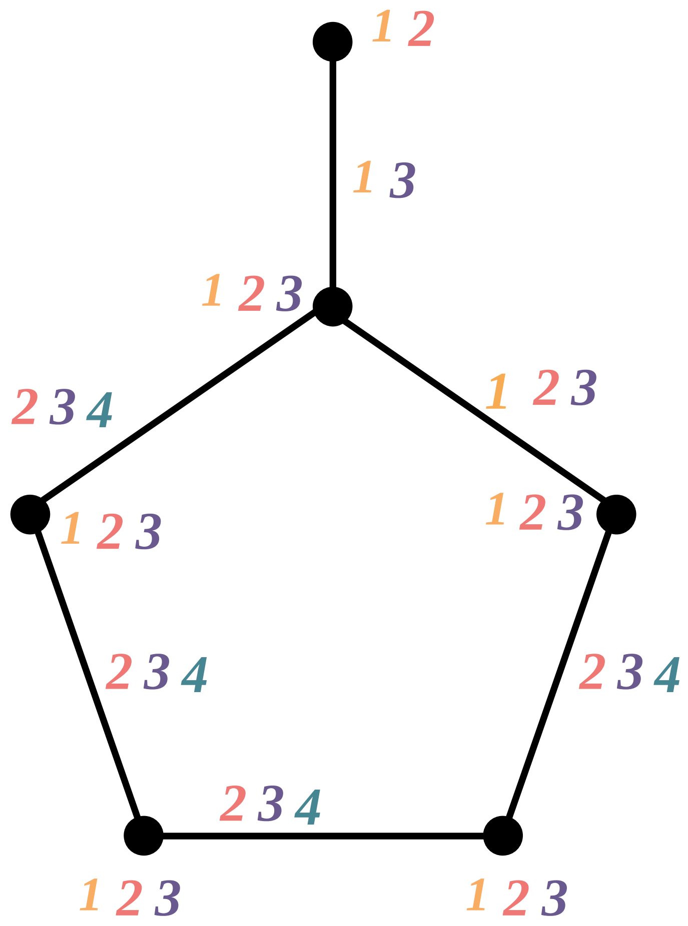
The List Total-Coloring of Graphs

Let G be a graph. For every $v \in V(G)$ ***and*** $e \in E(G)$***, L(v) and L(e) denote a list of colors available at v and a list of colors available for e, respectively. A list total-coloring is a proper total coloring f with f(v) chosen from L(v) for each v and f(e) chosen from L(e) for each e, respectively. The total-choosablity,*** $\chi_l''(G)$***, is the minimum k such that every assignment of lists of size k to every vertices and edges yields a proper list coloring and list edge-coloring for G.***

- ***For every graph G,*** $\chi_l''(G) \geq \Delta(G$

If G is a tree with $\Delta(G) \geq 2$
then $\chi_l''(G) = \Delta(G) + 1.$

Example.



What we have done…

1. The list chromatic number of (

   There is a conjecture that for every graph G,
   $\chi_l'(G)=\chi'(G).$

   For $l\geq 3,\ \chi_l(G)\leq 3.$

2. The list edge-chromatic number of (0 mod l)-cycle graphs

   Galvin (1995). For every (0 mod l)-cycle graph G,

   $$\chi_l'(G)=\chi'(G)=\Delta(G).$$

   $$\text{For } l\geq 3,\ \chi_l'(G)=$$

   It is not hard to see that for every bipartite graph G,
   $\chi_l''(G)=\Delta(G)+2.$

3. The list total-chromatic numb

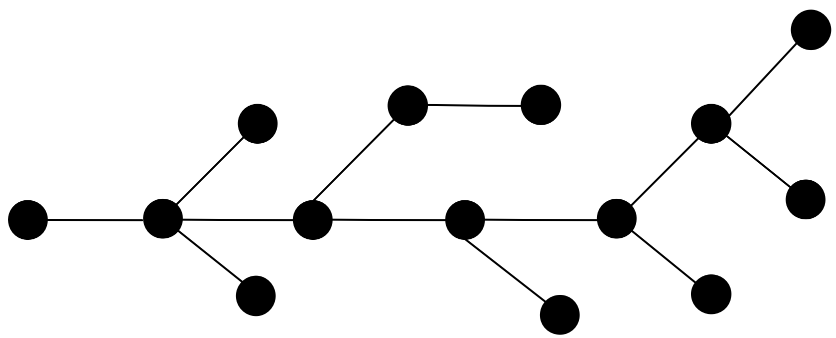
   If $l\notin\{1,2,4\}$ and $\Delta(G)\geq 3$, th $\chi_l(G)=\Delta(G)+1.$

Remark. ***For every natural number k, there is a bipartite graph G such that*** $\chi_l(G) > k$.

Theorem.* Let G be a graph and** $l \geq 3$ ***be a natural number. If G is a (0 mod l)-cycle graph, then $\chi_l(G) \leq 3.$

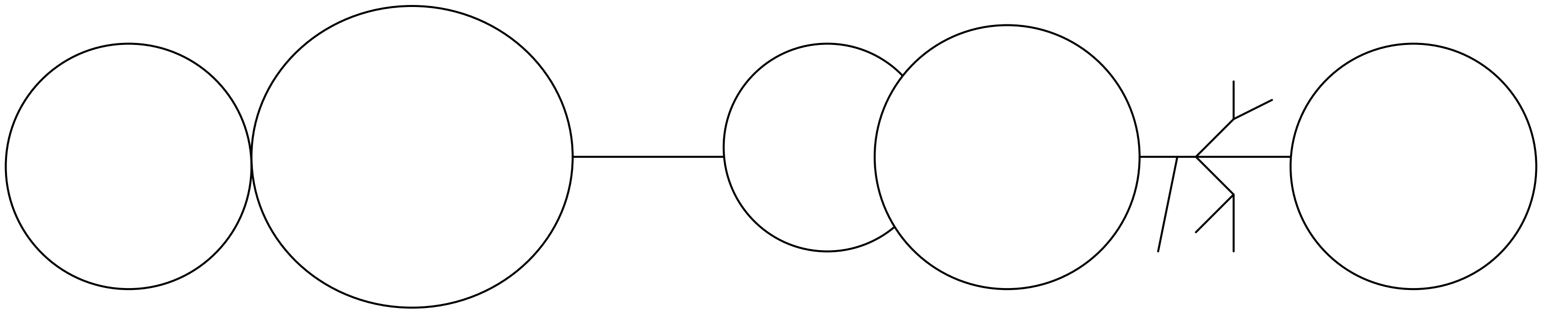
And so, $\chi(G) \leq \chi_l(G) \leq 3.$

For a positive integer s, a multigraph G is said to be s-degenerated if G can be reduced to K_1 by successive removal of vertices of degree at most s.



It is proved that for $l \geq 3$, *every (0 mod l)-cycle graph contains at least a vertex of degree 2.*

So, for every integer l, $l \geq 3$, *a (0 mod l)-cycle graph is 2-degenerated.*



It is easy to see that every s-degenerated graph is (s+1)-choosable.

So, it is concluded that for $l \geq 3$***, every (0 mod l)-cycle graph is 3-degenerated.***

For every graph G, the average degree of G is denoted by ad(G), where $ad(G)=\dfrac{2|E(G)|}{|V(G)|}$.

The maximum average degree, denoted by mad(G), is the maximum value of ad(H), where H is taken from all the subgraphs of G.

Theorem. For every s-degenerated graph G, $mad(G) < 2s$.

Theorem. ***Let k be an integer with*** $k \geq 4$***, and G be a graph with*** $mad(G) \leq k$ ***and*** $\Delta(G) \geq 0.5(k^2 - k + 2)$***, then*** $\chi_l'(G) = \Delta(G)$ ***and*** $\chi_l''(G) = \Delta(G) + 1$***.***

Corollary. ***Let l be an integer with*** $l \geq 3$ ***. Then every (0 mod l)-cycle graph G with*** $\Delta(G) \geq 7$ ***is*** $\Delta(G)$***-edge choosable and*** $(\Delta(G) + 1)$ ***-total choosable.***

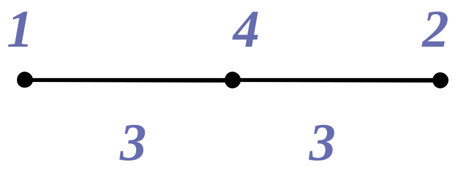
Theorem. ***For*** $l \geq 3$, ***except odd cycles and l = 4, every (0 mod l)-cycle graph G satisfies*** $\chi_l'(G) = \chi'(G) = \Delta(G)$.

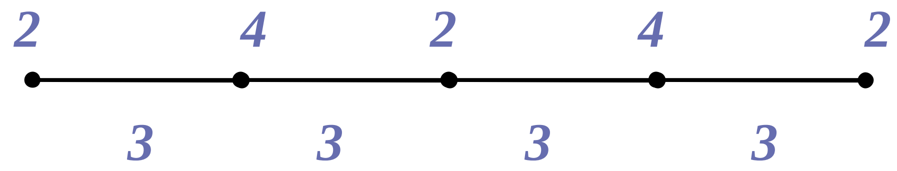
Theorem. ***Let l be a positive integer with*** $l \notin \{1,2,4\}$.
Then every (0 mod l)-cycle graph with $\Delta(G) \geq 3$,
is $(\Delta(G)+1)$***-total-choosable.***

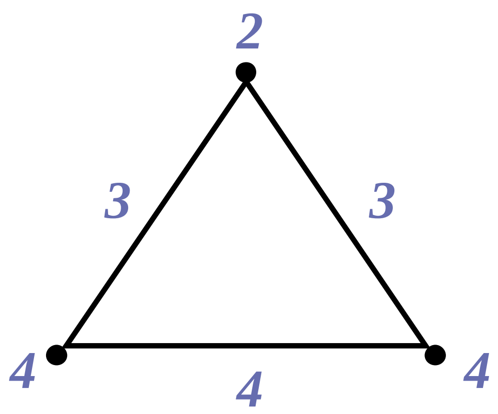
Some useful Lemmas for the proof of the theorem

$$\chi''(P_n) = 3, \qquad \chi''(C_n) = \begin{cases} 3 & n=3k \\ 4 & \text{otherwise} \end{cases}$$

Let H be a subgraph of G with $H \neq G$*. Then* $\chi_l''(H) \leq \Delta(G)+1$.







Conjecture.

Every (0 mod 4)-cycle graph with $3 \le \Delta(G) \le 6$ ***satisfies*** $\chi_l'(G) = \Delta(G)$ *and* $\chi_l''(G) = \Delta(G) + 1.$

Thanks for your attention