Vertex-Coloring Edge-Weighting

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Background

Let G be a graph, F be a field, and S be a subset of F:

An S-edge-weighting of G is an assignment of weights by the elements of S to each edge of G.

A k-edge-weighting of G is an assignment of an integer weight, $w(e) \in \{1, \ldots, k\}$ to each edge e.

c_v ***denotes the color of vertex v, that is***

the sum of the weights on the edges incident to v.

An edge weighting is called proper edge-weighting

if no two edges incident to the same vertex

receive the same weight.

An edge weighting is called a vertex-injective if for

every pair of distinct vertices, {u, v}, the colors

c_u and c_v are distinct.

An edge weighting is called a vertex-coloring if for every edge (u ,v), the colors c_v and c_u are distinct.

***Edge component:** It is a component which is isomorphic to K_2.*

The first question about non-proper edge weightings was introduced by ***Chartrand***, ***Jacobson***, ***Lehel***, ***Oellermann***, ***Ruiz*** and ***Saba*** in **1998**, asks for the smallest k such that G permits vertex-injective k-edge-weighting.

This graph parameter is denoted ***s(G).***

- ***For any graph G on $n \geq 4$ vertices we have $s(G) \leq n-1$.***

- ***For every r-regular graph G we have $s(G) \leq \frac{n}{r} + c$, for some constant c.***

1, 2, 3-Conjecture (2004)

Karonski, ***Luczak*** and ***Thomason*** initiated the study of vertex-coloring edge-weightings.

They conjectured that:

1, 2, 3- Conjecture.

Every graph without an edge component permits a

vertex coloring 3-edge weighting.

Theorem. ***Every graph G with no edge component and $\chi(G) \leq 3$ permits vertex coloring 3-edge weighting.***

Theorem. ***Every graph G with no edge component permits vertex coloring 213-edge weighting.***

Addario-Berry, ***Dalal, McDiarmid***, ***Reed*** and ***Thomason.***

(2007)

Theorem. ***Every graph G with no edge component***

permits vertex coloring 30-edge weighting.

Addario-Berry, ***Dalal*** and ***Reed.***

(2008)

Theorem. Every graph G without an edge component permits vertex coloring 16-edge weighting.

An edge weighting is called a vertex-distinguishing

if for every vertex u and v, the multiset of weights appearing on edges incident to u is distinct from the multiset of weights appearing on edges incident to v.

What is the smallest k such that G permits vertex-distinguishing k-edge weighting?

This graph parameter is denoted ***c(G).***

Theorem. If G is an r-regular graph, then there exist constants c_1 and c_2 such that $c_1 n^{\frac{1}{r}} \leq c(G) \leq c_2 n^{\frac{1}{r}}$

An edge weighting is called an adjacent vertex-distinguishing if for every edge (u,v), the multiset of weights appearing on edges incident to u is distinct from the multiset of weights appearing on edges incident to v.

Multiset version of the 1, 2, 3-conjecture

What is the minimum k such that there is an adjacent vertex distinguishing k-edge weighting?

Addario-Berry, ***Aldred***, ***Dalal*** and ***Reed*** :

Theorem. ***Every graph G without an edge component has an adjacent vertex distinguishing 4-edge weighting.***

Theorem. ***Every graph G without an edge component and with minimum degree at least 1000, has an adjacent vertex-distinguishing 3-edge-weighting.***

Our Results

Theorem. ***Let a, b and c are three distinct real numbers.***

i. ***For every natural number $n \geq 3$, the complete Graph K_n has a vertex coloring {a, b, c}-edge weighting.***

ii. ***For every natural numbers m and n with $m + n > 2$, the complete bipartite graph $K_{m,n}$ has a vertex coloring {a, b}-edge-weighting.***

Question. ***Suppose 1, 2, 3 conjecture is true for graph G. Does G permit vertex coloring {a, b, c}-edge weighting for every real distinct number a, b, and c?***

A dynamic coloring is a proper vertex k-coloring of G if for every vertex v with degree at least 2, the neighbors of v receive at least two different colors.

The dynamic chromatic number of G is the smallest integer k such that G has a k-dynamic coloring and denoted by $\chi_2(G)$.

Theorem. ***Let G be an r-regular bipartite graph where $r \geq 4$. Then there is a dynamic vertex coloring of G by 4 colors, using 2 colors in each color class.***

Theorem. ***For each natural numbers n and r and two distinct real numbers a and b with r ≥ 4, every r-regular bipartite graph has a vertex coloring {a, b}-edge-weighting.***

Proof.

Let X = $\{v_1, \ldots, v_n\}$ and Y = $\{u_1, \ldots, u_n\}$ be two parts of G. Without loss of generality, in part X, by previous theorem there exists a dynamic coloring with two colors a and b. Now, we color all edges incident with a vertex with color a by a, and a vertex with color b by b.

So, in part Y, for every k, $1 \leq k \leq n$, we have

$c_{u_k} = a s_k + (r - s_k) b$, for some natural number $1 \leq s_k \leq r\text{-}1$.

Now, if $s_k a + (r - s_k) b = ra$ or $s_k a + (r - s_k) b = rb$.

Then, we have $r = s_k$ or $a = b$, respectively.

But this is contradiction. So we have $s_k a + (r - s_k) b \neq ra$ and $s_k a + (r - s_k) b \neq rb$. Thus, we get the result.

Conjecture. ***Every 3-regular bipartite graph***
has a vertex coloring {a, b}-edge-weighting.

Let F be a field. An edge weighting is called a vertex colorable k-list edge-weighting if for every $e \in E(G)$, and every list assignment, $L_e \subseteq F$, $|L_e| = k$, to edge e, one could obtain a vertex coloring edge-weighting such that $w(e) \in L_e$.

An edge weighting is called a vertex colorable positive k-list-edge-weighting if in the previous definition we change F to positive real numbers.

Theorem. ***Every tree with $n \geq 3$ is vertex colorable positive 2-list-edge weighting.***

If edge e and vertex i are incident then we write e ~ i.

For every g = ij ∈ $E(G)$ (i < j) assign the variable x_g to g.

Define $f_g(x_{e_1},\dots, x_{e_n}) = \sum_{e\in E(G), e\sim i} x_e - \sum_{e\in E(G), e\sim i} x_e$

Now, we introduce $\theta_G(x_{e_1},\dots, x_{e_m})$ as $\prod_{e\in E} f_e(x_{e_1},\dots, x_{e_n})$.

For every $(a_1, \dots, a_m) \in R^m$.

we associate an edge weighting w, such that w(e_i) $= a_i$.

for every i, 1 ≤ i ≤ m.

It is easy to see that $\theta_G(a_1,\dots, a_m) \neq 0$ if and only if the edge weighting corresponding to $(a_1,\dots, a_m)$ forms a vertex-coloring edge-weighting.

Theorem. (Combinatorial Nullstellensatz).

Let F be an arbitrary field, and let f = f(x_1 , . . . , x_n) be a polynomial in F[x_1 , . . . , x_n]. Suppose that the degree of f is $\sum_{i=1}^{n} t_i$ where each t_i is a nonnegative integer, and suppose the coefficient of $\prod_{i=1}^{n} x_i^{t_i}$ in f is nonzero. Then, if S_1 , . . . , S_n are subset of F with | S_i | > t_i , there are $s_1 \in S_1$, . . . , $s_n \in S_n$ so that f(s_1 , . . . , s_n) $\neq 0$.

Theorem. ***Let G be a graph and $\Delta(G) \leq 3$. Then G is vertex colorable 5-list edge-weighting.***

Proof. Let E(G) = $\{e_1, \ldots, e_m\}$. First note that for every two edges $e, e' \in E(G)$, x_e has nonzero coefficient in $F_{e_i}(x_{e_1}, \ldots, x_e)$ if and only if e and e' are incident. Since $\Delta(G) \leq 3$, each edge of G is incident with at most 4 other edges. Thus, for each e $\in E(G)$, the variable x_e has nonzero coefficient in at most four $F_{e_i}(x_{e_1}, \ldots, x_{e_m})$.

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Now, in each monomial of $\theta_G(x_{e_1}, \ldots, x_{e_m})$ every variable x_e has degree at most 4. Since $\theta_G(x_{e_1}, \ldots, x_{e_m})$ is nonzero, by Theorem 2 if we assign a list of five numbers to each edge, then we can weight each edges with a number from its list so that it induces a vertex-coloring edge-weighting.

Thanks for your attention

Proof.

i. We apply by induction on n. Without loss of generality assume that a < b < c. For n = 3, the assertion is trivial. Assume that K_{n-1} has a vertex coloring {a, b, c}-edge weighting. Now, add a new vertex v_n to the V(K_{n-1}) and join it to all vertices of V(K_{n-1}).

If there exists a vertex $v_i \in$V(K_{n-1}) such that all edges incident with v_i have weights c, then we give weight a to all edges incident with v_n . Since c_{v_n} is (n-1) a, and for every j, 1 $\leq$ j $\leq$ n-1, c_{v_j} is more than (n-1) a, we obtain the result.

Thus, suppose that there is no vertex in V(K_{n-1}) such that all incident edges have the same weight c. In this case we assign c to all edges incident with v_n . Since c_{v_n} is (n-1) c, and for every j, 1 $\le$ j $\le$ n-1, c_{v_j} < (n-1) c, we obtain the result.

ii. Let X = $\{v_1, \ldots, v_n\}$ and Y = $\{u_1, \ldots, u_m\}$ be two parts of $K_{m,n}$. Without loss of generality, in part X, for every i, $1 \leq i \leq l$, and $1 \leq l \leq n - 1$, we give weight a to all edges incident with v_i, and for every j, $l < j \leq n$, we give weight b ($b \neq a$) to all edges incident with v_j.

So, in part Y, for every k, $1 \leq k \leq m$, we have $c_{u_k} = la + (n - l)b$. Let m , n ≤ 2. Then, we have P_3 or C_4.

We give weight a and b to two edges of P_3 , and a, b, b, and a, to edges of C_4 and we obtain the result.
Thus, Without loss of generality suppose that $n > 2$.
Then, we have $la + (n - 1)b \neq ma, mb$ for some positive integer l. Therefore this vertex coloring is proper.