On the sum of Laplacian eigenvalues of a graph

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Laplacian eigenvalues

$G = (V, E)$: a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$

Laplacian matrix $L(G)$:

$$L(G)_{ij} = \begin{cases} \deg(v_i) & \text{if} \quad i = j, \\ -1 & \text{if} \quad \{i, j\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

The matrix $L(G)$ is positive semidefinite, so its eigenvalues are real and non-negative:

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0.$$

The Grone-Merris conjecture (1994)

$d_1 \geq d_2 \geq \cdots \geq d_n$: vertex degrees of graph G

Conjugate or *dual* degrees:

$$d'_i = |\{x | d_x \geq i\}|$$

Then

$$\mu_1 + \mu_2 + \cdots + \mu_k \leq \sum_{i=1}^{k} d'_i.$$

Brouwer's conjecture (2007)

$$\mu_1 + \mu_2 + \cdots + \mu_k \leq e + \binom{k+1}{2}.$$

Theorem (Rojo *et al.*, 2000)

$$\mu_1 + \mu_2 + \cdots + \mu_k \leq \frac{2ek + \sqrt{ek(n-k-1)(n^2-n-2e)}}{n-1}.$$

Theorem (Fan, 1949)

Let A and B be symmetric matrices of order n. Then

$$\sum_{i=1}^{k} \lambda_i(A+B) \leq \sum_{i=1}^{k} \lambda_i(A) + \sum_{i=1}^{k} \lambda_i(B).$$

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Corollary

Let G and H be two edge disjoint graphs on the same vertex set V. Then

$$\sum_{i=1}^{k} \mu_i(G \cup H) \leq \sum_{i=1}^{k} \mu_i(G) + \sum_{i=1}^{k} \lambda_i(H).$$

The case $k = 2$

Brouwer's conjecture:

$$\mu_1 + \mu_2 \leq e + 3.$$

Let M be a set of four disjoint edges in G. Then

$$\mu_1(G) + \mu_2(G) \leq \mu_1(G \setminus M) + \mu_2(G \setminus M) + 4.$$

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So for $k = 2$ it suffices to consider graphs with matching number at most 3.

Graphs with matching number 1

$$K_{1,m-1} + (n-m)K_1$$

$$K_3 + (n-3)K_1$$

Graphs with matching number 2

We consider two cases.

(i) G has a triangle. Then it follows from the Grone-Merris bound or a decomposition.

(ii) G has no triangle. Then it follows from the Grone-Merris bound or the following lemma.

Lemma. *Let G be a subgraph of $K_{2,m}$. Then $\mu_1 + \mu_2 \leq e + 3$.*