A Conjecture on

Circular Chromatic Number

R. Tusserkani

<u>History & Acknowledgement</u>

- 1988 ***Andrew Vince*** *introduced the notion of **<u>star</u>** chromatic number.*
- *Journal of Graph Theory Supported the New Notion.*
- 1990 ***Pavol Hell*** Proved that Computing the **<u>circular</u>** chromatic number of a graph is NP-Hard.
- 1996 ***Xuding Zhu*** started writing his survey.

Circular chromatic number: a survey

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First draft, March, 1997; revised Nov., 1997, and Nov., 1998

Abstract

The circular chromatic number $\chi_c(G)$ of a graph G (also known as "the star-chromatic number"), is a natural generalization of the chromatic number of a graph. In this paper, we survey results on this topic, concentrating on the relations among the circular chromatic number, the chromatic number and some other parameters of a graph. Some of the results and/or proofs presented here are new. The last section is devoted to open problems. We pose 28 open problems, and discuss partial results and give references (if any) for each of these problems.

28 open problems

Abstract

The circular chromatic number $\chi_c(G)$ of a graph G (also known as "the star-chromatic number"), is a natural generalization of the chromatic number of a graph. In this paper, we survey results on this topic, concentrating on the relations among the circular chromatic number, the chromatic number and some other parameters of a graph. Some of the results and/or proofs presented here are new. The last section is devoted to open problems. We pose 28 open problems, and discuss partial results and give references (if any) for each of these problems.

Fast Breaking Comment by Professor Xuding Zhu - Microsoft Internet Explorer

Address http://esi-topics.com/fbp/comments/december02-ProfessorXudingZhu.html

Professor Xuding Zhu answers a few questions about this month's fast breaking paper in the field of Mathematics.

From • >>December 2002

Field: Mathematics

Article Title: "Circular chromatic number: a survey"

Authors: Zhu, XD
Journal: DISCRETE MATH
Volume: 229
Page: 371-410
Year: FEB 28 2001
* Natl Sun Yat Sen Univ, Dept Appl Math, Kaohsiung 80424, Taiwan.
* Natl Sun Yat Sen Univ, Dept Appl Math, Kaohsiung 80424, Taiwan.

ST: Why do you think your paper is highly cited?

The circular chromatic number is a new concept in graph theory. More and more people working on graph coloring and flow problems are getting interested in this concept.

ST: Does it describe a new discovery or a new methodology that's useful to others?

This paper reviews current research results on this topic, and it also contains a list of open problems. It puts the research results in a unified framework, and it is a convenient source of information on this subject.

ST: Could you summarize the significance of your paper in layman's terms?

Circular coloring of a graph colors the vertices of the graph by points of a circle in such a way that adjacent vertices are colored by points that are far apart in the circle. It is a variation of the traditional vertex coloring of graphs, in which the vertices are colored by integers, and colors of adjacent vertices are distinct. The circular chromatic number is a refinement of the chromatic number of a graph, and reveals more

Done Internet

Definitions & Preliminaries

- A (k,d)-coloring of a graph assigns congruence classes mod k to the vertices so that adjacent vertices are assigned classes that are at least d apart.

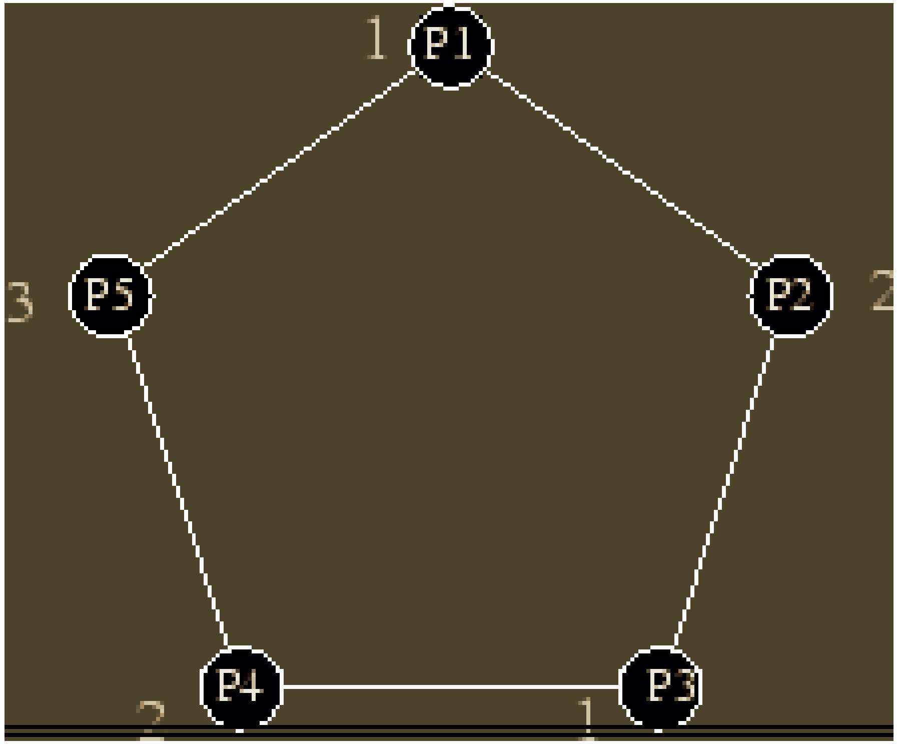
- *(k,1)-coloring* **is just an ordinary** *k-coloring***.**

Which *(k,d)*-coloring is *better*

- The ***"best"*** *(k,d)-coloring is that one in which k/d is as small as possible.*

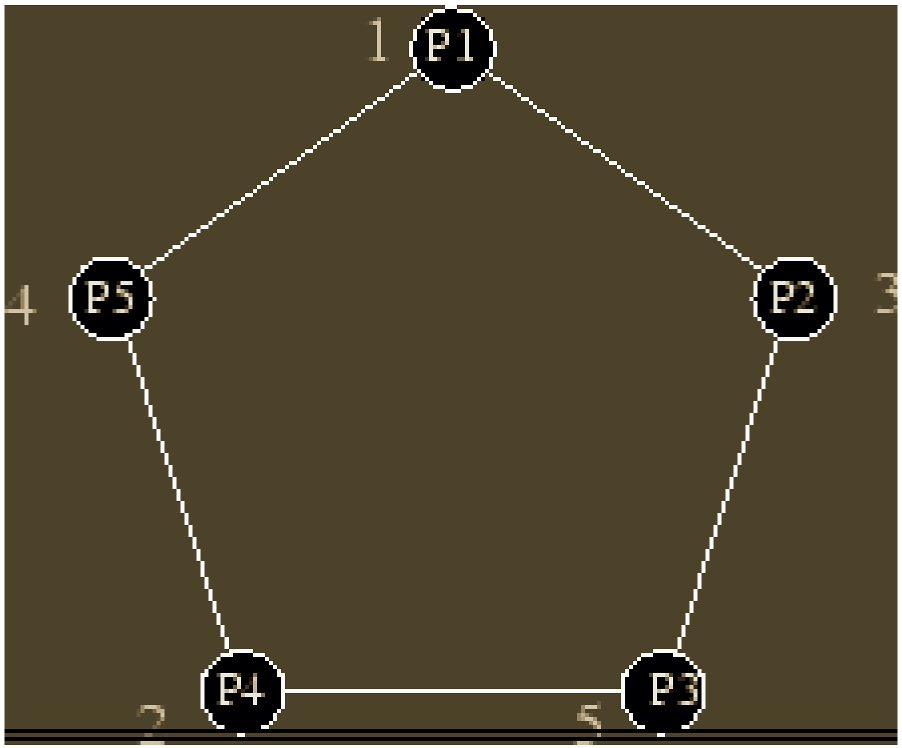
The best *(k,d)-coloring* for C_5?

(3,1)-coloring



The best *(k,d)-coloring* for C_5?

(5,2)-coloring



circular chromatic number

- **The** *circular chromatic number* **of G is the infimum of k/d such that G has a (k,d)-coloring.**

Vince Theorem:

- The ***infimum*** in the definition of circular chromatic number is always attained.

- The circular chromatic number is rational.

Known Results:

The following hold for any graph G:

$$\chi_f(G) \le \chi_c(G) \le \chi(G)$$

$$\lceil \chi_c(G) \rceil = \chi(G)$$

$$\chi_c(C_{2n+1}) = 2 + \frac{1}{n} \quad \rightarrow \mathbf{2}$$

$$\chi_c(C_{2n}) = \chi(C_{2n}) = 2$$

$$\chi_c(\mathit{Peterson}) = 3.$$

1. Complexity?

Question 8.23 *what is the complexity of determining whether or not* $\chi_c(G) = \chi(G)$, *if the chromatic number* $\chi(G)$ *is known?*

Our answer:

- If the chromatic number is known, the problem of determining whether or not $\chi_c = \chi$ is still NP-Hard.

On the Complexity of the Circular Chromatic Number

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Abstract: Circular chromatic number, χ_c is a natural generalization of chromatic number. It is known that it is **NP**-hard to determine whether or not an arbitrary graph G satisfies $\chi(G) = \chi_c(G)$. In this paper we prove that this problem is **NP**-hard even if the chromatic number of the graph is known. This answers a question of Xuding Zhu. Also we prove that for all positive integers $k \geq 2$ and $n \geq 3$, for a given graph G with $\chi(G) = n$, it is **NP**-complete to verify if $\chi_c(G) \leq n - \frac{1}{k}$. J Graph Theory 00: 1–5, 2004

2. Circular Edge Chromatic Number?

- **Definition.** Circular Edge Chromatic Number of a graph is Circular Chromatic Number of its line graph.

The Questions:

Question 8.3 *Prove that if G is a 2-edge connected cubic planar graph, and $H = L(G)$ is the line graph of G, then $\chi_c(H) < 4$, without using the Four Color Theorem.*

Indeed, we do not know any 2-edge connected cubic graph whose line graph has circular chromatic number 4. One might expect that the line graph $L(P)$ of the Petersen graph P has circular chromatic number 4 (why not?). However, this is not the case. The graph $L(P)$ has circular chromatic number $11/3$.

Question 8.4 *Are there any 2-edge connected cubic graph G whose line graph has circular chromatic number 4 ?*

Circular Edge Chromatic Number of Cubic Graphs

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Abstract

Circular edge chromatic number of a graph is defined as the circular chromatic number of its line graph. In this paper we solve a problem of X. Zhu by proving that the circular edge chromatic number of every 2-edge connected cubic graph is less than or equal to $11/3$.

Circular chromatic index of graphs of maximum degree 3

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Ruzbeh Tusserkani,[¶] and Xuding Zhu[‖**]

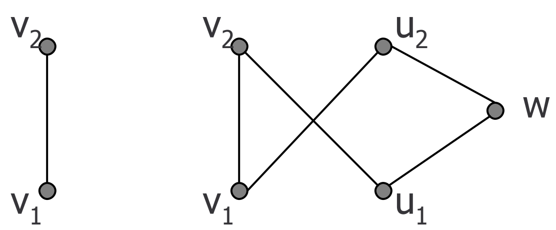
February 13, 2004

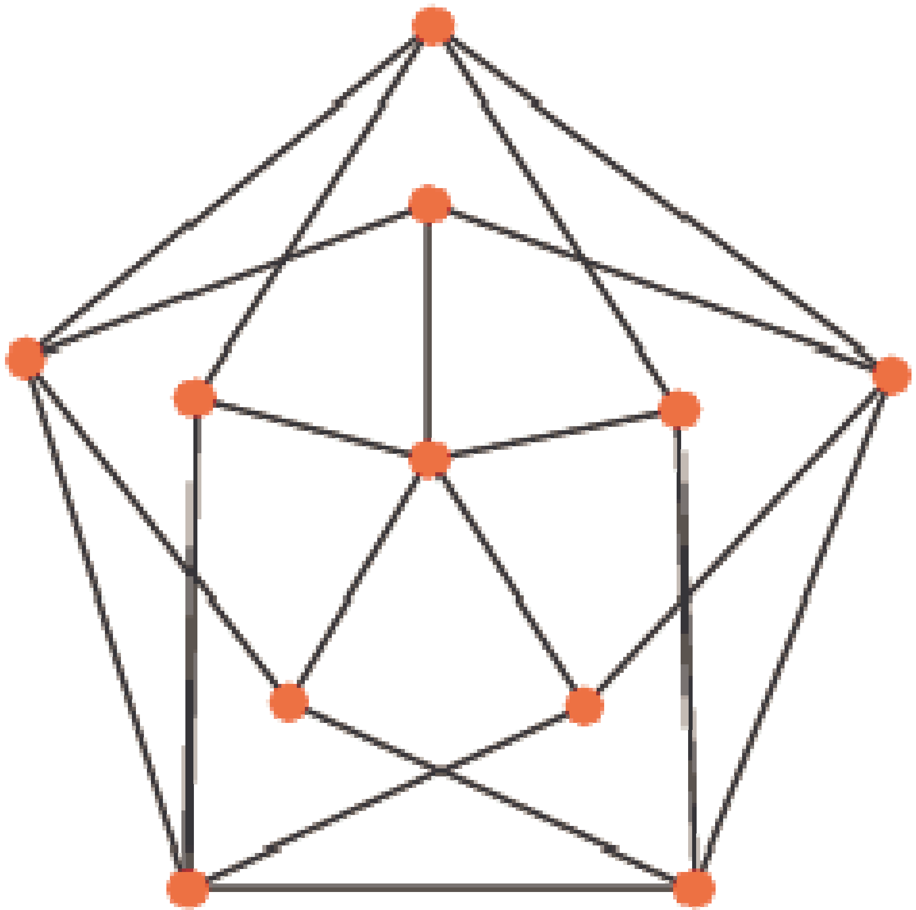
Abstract

This paper proves that if G is a graph (parallel edges allowed) of maximum degree 3, then $\chi'_c(G) \leq 11/3$ provided that G does not contain H_1 or H_2 as a subgraph, where H_1 and H_2 are obtained by subdividing one edge of K_2^3 (the graph with three parallel edges between two vertices) and K_4, respectively. As $\chi'_c(H_1) = \chi'_c(H_2) = 4$, our result implies that there is no graph G with $11/3 < \chi'_c(G) < 4$. It also implies that if G is a 2-edge connected cubic graph, then $\chi'(G) \leq 11/3$.

3. Circular Chromatic Number of Mycielski's Graph?

- Definition:
  - A simple graph G, **Mycielski's construction** produces a simple graph M(G) containing G. Beginning with G having vertex $\{v_1, v_2, \ldots, v_n\}$, add $U=\{u_1, u_2, \ldots, u_n\}$ and w. add edges to make u_i to adjacent to $N_G(v_i)$, and finally let N(w)=U.





Circular chromatic numbers of Mycielski's graphs

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May 1996 (revisions November 1997, June 1998, December 1998)

Abstract

In a search for triangle-free graphs with arbitrarily large chromatic numbers, Mycielski developed a graph transformation that transforms a graph G into a new graph $\mu(G)$, we now call the Mycielskian of G, which has the same clique number as G and whose chromatic number equals $\chi(G)+1$. Let $\mu^n(G) = \mu(\mu^{n-1}(G))$ for $n \geq 2$. This paper investigates the circular chromatic numbers of Mycielski's graphs. In particular, the following results are proved in this paper: (1) for any graph G of chromatic number n, $\chi_c(\mu^{n-1}(G)) \leq \chi(\mu^{n-1}(G)) - \frac{1}{2}$; (2) if a graph G satisfies $\chi_c(G) \leq \chi(G) - \frac{1}{d}$ with $d = 2$ or 3, then $\chi_c(\mu^2(G)) \leq \chi(\mu^2(G)) - \frac{1}{d}$; (3) for any graph G of chromatic number 3, $\chi_c(\mu(G)) = \chi(\mu(G)) = 4$; (4) $\chi_c(\mu(K_n)) = \chi(\mu(K_n)) = n+1$ for $n \geq 3$ and $\chi_c(\mu^2(K_n)) = \chi(\mu^2(K_n)) = n+2$ for $n \geq 4$.

Conjecture:

(Chang, Huang, and Zhu 1999)

$$\text{for all } n \geq t+2,$$

$$\chi_c(M^t(K_n)) = \chi(M^t(K_n)) = n+t$$

Circular chromatic number and Mycielski construction

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Jan, 2001

Abstract

This paper gives a sufficient condition for a graph G to have its circular chromatic number equal its chromatic number. By using this result, we prove that for any integer $t \geq 1$, there exists an integer n such that for all $k \geq n$ $\chi_c(M^t(K_k)) = \chi(M^t(K_k))$.

Circular Chromatic Number for Iterated Mycielski Graphs *

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Abstract

For a graph G, let $\mathrm{M}(G)$ denote the Mycielski graph of G. The t-th iterated Mycielski graph of G, $\mathrm{M}^t(G)$, is defined recursively by $\mathrm{M}^0(G) = G$, and $\mathrm{M}^t(G) = \mathrm{M}(\mathrm{M}^{t-1}(G))$ for $t \geq 1$. Let $\chi_c(G)$ denote the circular chromatic number of G. We prove two main results: 1) If G has a universal vertex x, then $\chi_c(\mathrm{M}(G)) = \chi(\mathrm{M}(G))$ if $\chi_c(G-x) > \chi(G) - 1/2$ and G is not a star, otherwise $\chi_c(\mathrm{M}(G)) = \chi(\mathrm{M}(G)) - 1/2$; and 2) $\chi_c(\mathrm{M}^t(K_m)) = \chi(\mathrm{M}^t(K_m))$ if $m \geq 2^{t-1} + 2t - 2$ and $t \geq 2$.

$$\chi_c(\mathrm{M}^t(K_m)) = \chi(\mathrm{M}^t(K_m)) \text{ if } m \geq 2^{t-1} + 2t - 2$$

The Circular Chromatic Number of the Mycielski's graph $M^t(K_n)$ *

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Abstract

As a natural generalization of chromatic number of a graph, the circular chromatic number of graphs (or the star chromatic number) was introduced by A.Vince in 1988. Let $M^t(G)$ denote the tth iterated Mycielski graph of G. It was conjectured by Chang, Huang and Zhu(Discrete mathematics,205(1999), 23-37) that for all $n \geq t+2$, $\chi_c(M^t(K_n)) = \chi(M^t(K_n)) = n+t$. In 2004, D.D.F. Liu proved the conjecture when $t \geq 2$, $n \geq 2^{t-1}+2t-2$. In this paper,we show that the result can be strengthened to the following: if $t \geq 4$, $n \geq \frac{11}{12}2^{t-1}+2t+\frac{1}{3}$, then $\chi_c(M^t(K_n)) = \chi(M^t(K_n))$.

if $t \geq 4$, $n \geq \frac{11}{12}2^{t-1} + 2t + \frac{1}{3}$, then $\chi_c(M^t(K_n)) = \chi(M^t(K_n))$.

Local chromatic number, Ky Fan's theorem, and circular colorings

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Abstract

The local chromatic number of a graph was introduced in [12]. It is in between the chromatic and fractional chromatic numbers. This motivates the study of the local chromatic number of graphs for which these quantities are far apart. Such graphs include Kneser graphs, their vertex color-critical subgraphs, the Schrijver (or stable Kneser) graphs; Mycielski graphs, and their generalizations; and Borsuk graphs. We give more or less tight bounds for the local chromatic number of many of these graphs.

We use an old topological result of Ky Fan [14] which generalizes the Borsuk-Ulam theorem. It implies the existence of a multicolored copy of the complete bipartite graph $K_{\lceil t/2\rceil,\lfloor t/2\rfloor}$ in every proper coloring of many graphs whose chromatic number t is determined via a topological argument. (This was in particular noted for Kneser graphs by Ky Fan [15].) This yields a lower bound of $\lceil t/2\rceil + 1$ for the local chromatic number of these graphs. We show this bound to be tight or almost tight in many cases.

As another consequence of the above we prove that the graphs considered here have equal circular and ordinary chromatic numbers if the latter is even. This partially proves a conjecture of Johnson, Holroyd, and Stahl and was independently attained by F. Meunier [38]. We also show that odd chromatic Schrijver graphs behave differently, their circular chromatic number can be arbitrarily close to the other extreme.

conjecture holds when $t + n$ is even.

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CIRCULAR CHROMATIC NUMBER AND MYCIELSKI GRAPHS

GENGHUA FAN*

As a natural generalization of graph coloring, Vince introduced the star chromatic number of a graph G and denoted it by $\chi^*(G)$. Later, Zhu called it circular chromatic number and denoted it by $\chi_c(G)$. Let $\chi(G)$ be the chromatic number of G. In this paper, it is shown that if the complement of G is non-hamiltonian, then $\chi_c(G)=\chi(G)$. Denote by $M(G)$ the Mycielski graph of G. Recursively define $M^m(G)=M(M^{m-1}(G))$. It was conjectured that if $m \leq n-2$, then $\chi_c(M^m(K_n))=\chi(M^m(K_n))$. Suppose that G is a graph on n vertices. We prove that if $\chi(G) \geq \frac{n+3}{2}$, then $\chi_c(M(G))=\chi(M(G))$. Let S be the set of vertices of degree $n-1$ in G. It is proved that if $|S| \geq 3$, then $\chi_c(M(G))=\chi(M(G))$, and if $|S| \geq 5$, then $\chi_c(M^2(G))=\chi(M^2(G))$, which implies the known results of Chang, Huang, and Zhu that if $n \geq 3$, $\chi_c(M(K_n))=\chi(M(K_n))$, and if $n \geq 5$, then $\chi_c(M^2(K_n))=\chi(M^2(K_n))$.

if $\chi(G) \geq \frac{n+3}{2}$, then $\chi_c(M(G)) = \chi(M(G))$.

An Upper Bound for the Circular Chromatic Number of Mycielski Graphs*

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Abstract

In a search for triangle-free graphs with arbitrarily large chromatic numbers, Mycielski ([15]) developed a graph transformation that transforms a graph G into a new graph M(G), we now call the Mycielskian of G. For $t \geq 2$, $M^t(G) = M(M^{t-1}(G))$. *Lam et al ([12]) generalized the definition to get the m-Mycielskian graph* ($\mu_m(G)$). *The problem of determining the circular chromatic numbers of these graphs has been investigated in many papers. In this paper, we shall study the range of* $\chi_c(M^t(G))$, *especially for* $G=K_n$ *or* K_d^k. *These investigations lead to an upper bound for the Mycielskians: if* $\chi_c(G) \leq (\chi(G)-1)+1/3$, *then* $\chi_c(M^{2t}(G)) \leq (\chi(M^{2t}(G))-1)+1/3$ *for every positive integer t.*

A note on circular chromatic number of Mycielski's graphs:

- The circular chromatic number of Mycielski's graph of every uniquely colorable graph is equal to the chromatic number of its Mycielski's graph.

A note on circular chromatic number of Mycielski's graphs

P. Afshani[*], H. Hatami[†], R. Tusserkani[‡]

Abstract

Circular chromatic number, χ_c is a natural generalization of chromatic number. Let $M(G)$ denotes the Mycielski's graph of G. Towards answering the question that for which graphs G, the circular chromatic number of $M(G)$ is equal to the chromatic number of $M(G)$, we prove that if in every χ-coloring of a graph, all the χ colors occur on every vertex together with its neighbor vertices, then this equality holds.

As a corollary we conclude that the circular chromatic number of Mycielski's graph of every uniquely colorable graph is equal to the chromatic number of its Mycielski's graph.

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Circular Chromatic Number
&
Lonely Runner Conjecture

Overview:

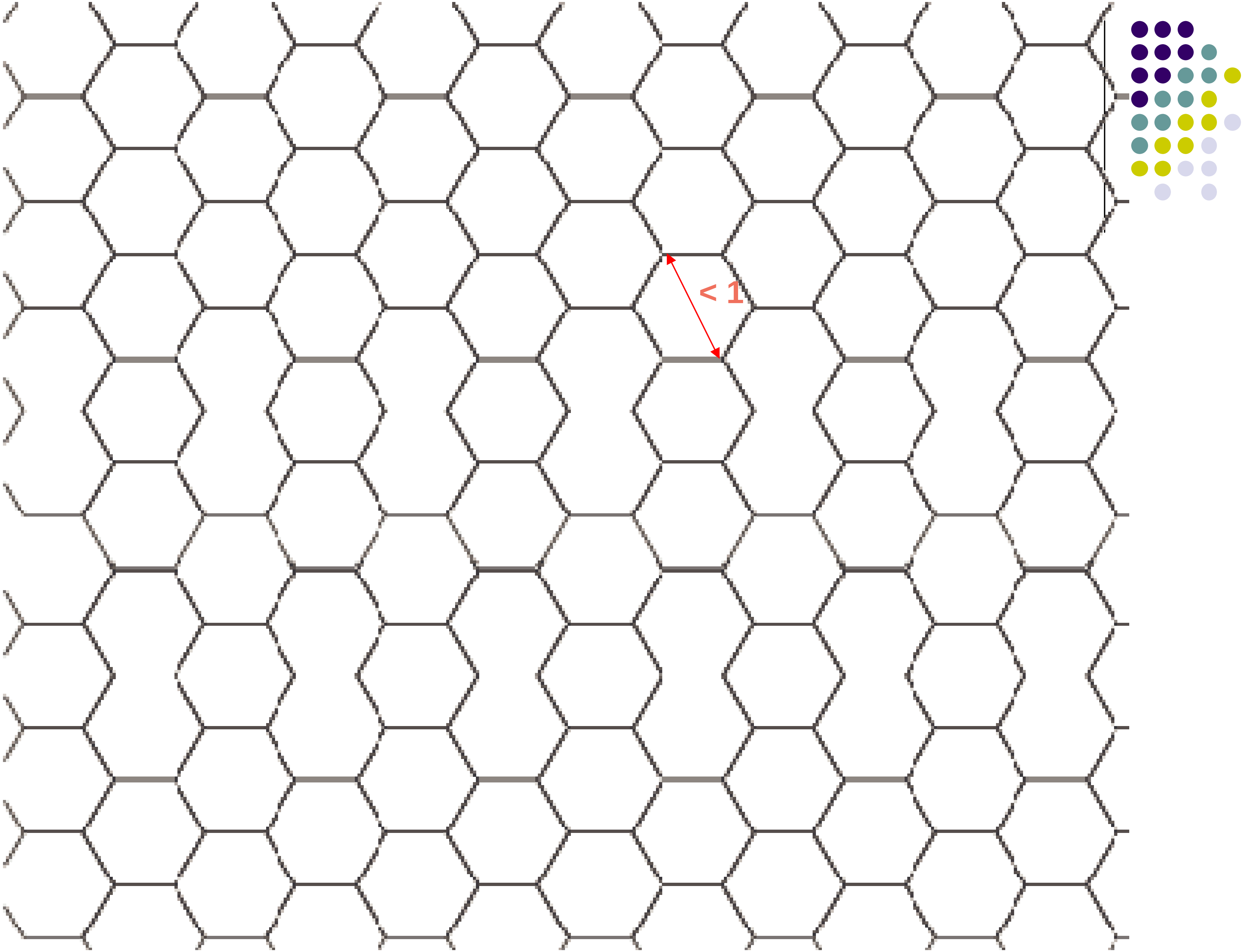
Plane coloring

↓

Distance Graphs → Fractional Chromatic Number → Circular Chromatic Number → Lonely Runner Conjecture

Plane Coloring Problem
(Edward Nelson 1950)

- Color all the points on the xy-plane so that any two points of unit distance apart get different colors.
- What is the smallest number of colors needed to accomplish the above ?
- Seven colors are enough [Moser & Moser, 1968]



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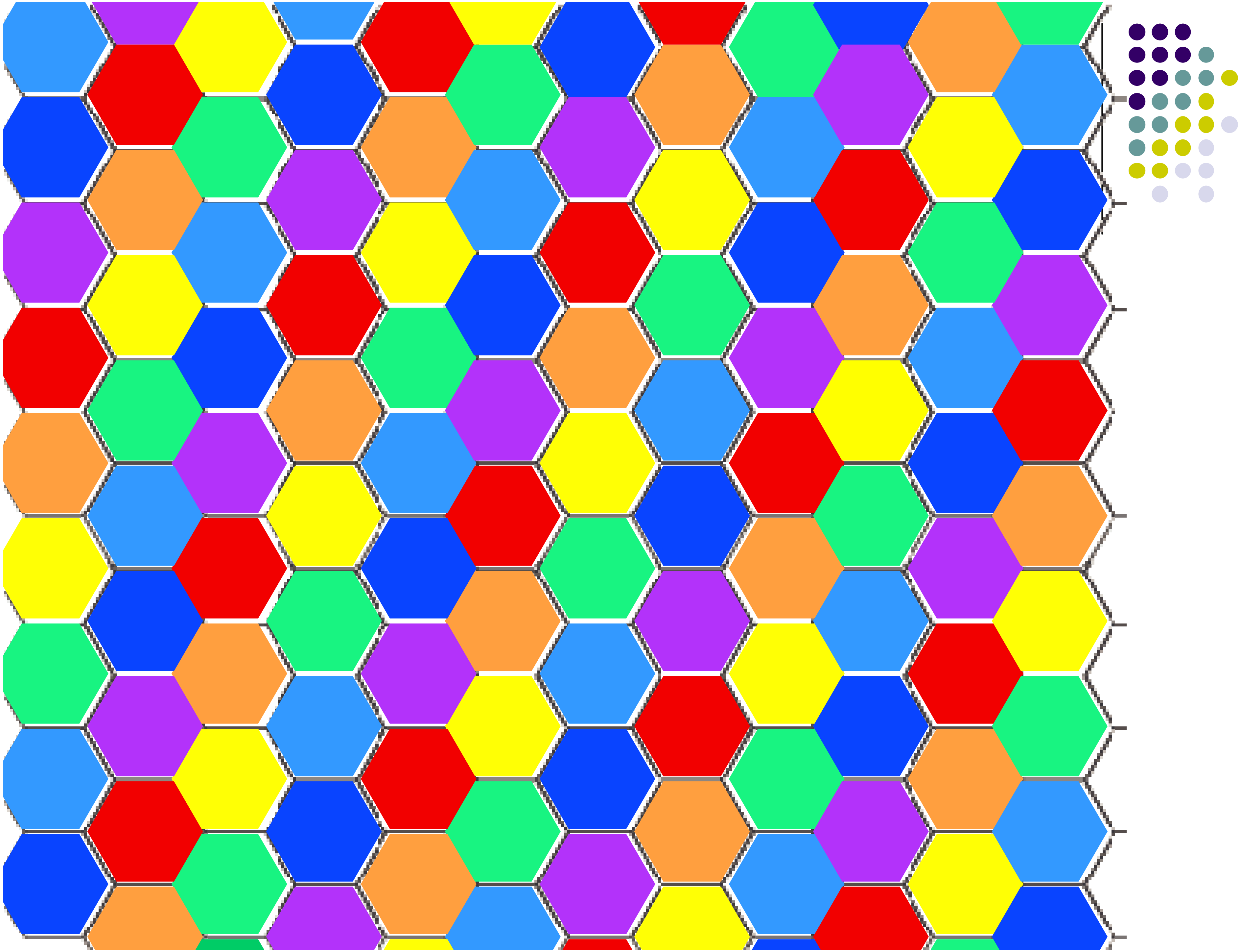
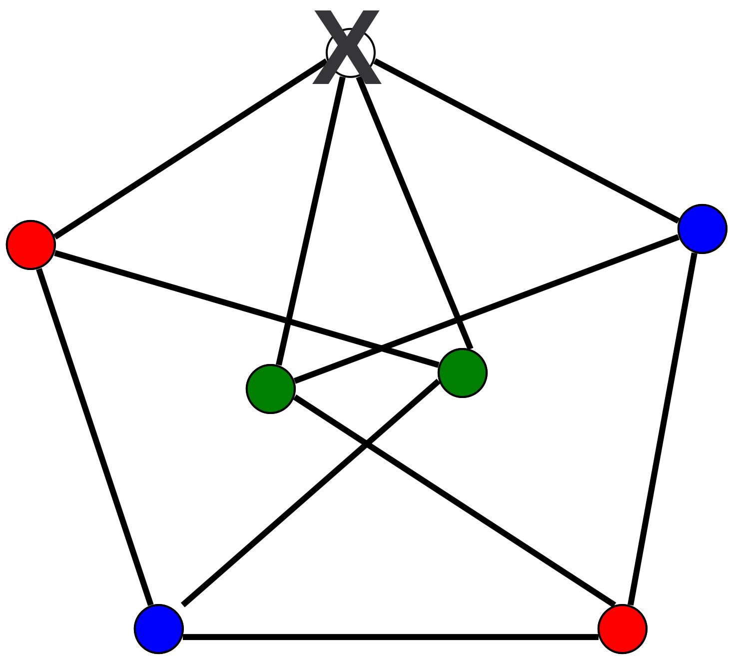


At least we need four colors for coloring the plane

Assume three colors, red, blue and green, are used.



Moser Spindle

Distance Graphs

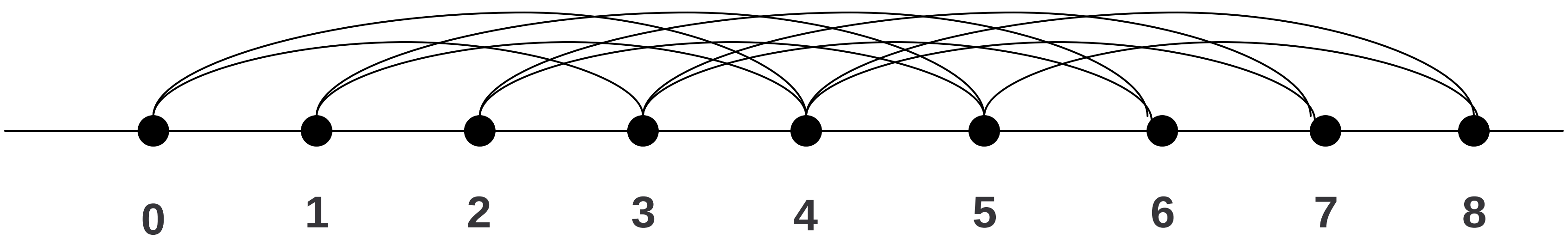
Eggleton, Erdős et. al. [1985 – 1987]

- For a given set D of positive integers, the distance graph G(Z, D) has:

    Vertices: All integers Z as vertices;

    Edges: u and v are adjacent ↔ |u - v| є D

D = {1, 3, 4}



Lonely Runner Conjecture

- Suppose k runners running on a circular field of circumference r. Suppose each runner keeps a constant speed and all runners have different speeds. A runner is called “lonely” at some moment if he or she has (circular) distance at least r/k apart from all other runners.
- Conjecture: For each runner, there exists some time that he or she is lonely.

The lonely runner with seven runners

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Mathematics Subject Classification: 11B75, 11J71, 05C15

Abstract

Suppose $k+1$ runners having nonzero constant pairwise distinct speeds run laps on a unit-length circular track starting at the same time and place. A runner is said to be lonely if she is at distance at least $1/(k+1)$ along the track to every other runner. The lonely runner conjecture states that every runner gets lonely. The conjecture has been proved up to six runners ($k \leq 5$). A formulation of the problem is related to the regular chromatic number of distance graphs. We use a new tool developed in this context to solve the first open case of the conjecture with seven runners.

Parameter involved in the Lonely Runner Conjecture

For any real x, let $\| x \|$ denote the shortest distance from x to an integer.

For instance, $\|3.2\| = 0.2$ and $\|4.9\|=0.1$.

Let D be a set of real numbers, let t be any real number:

$$\|D\,t\| := \min\{\, \|d\,t\| : d \in D\}.$$

$$\varphi(D) := \sup\{\, \|D\,t\| : t \in R\}.$$

Example

- $D = \{1, 3, 4\}$ (Four runners)

$$\|(1/3)\,D\| = \min\{1/3, 0, 1/3\} = 0$$

$$\|(1/4)\,D\| = \min\{1/4, 1/4, 0\} = 0$$

$$\|(1/7)\,D\| = \min\{1/7, 3/7, 3/7\} = 1/7$$

$$\|(2/7)\,D\| = \min\{2/7, 1/7, 1/7\} = 1/7$$

$$\|(3/7)\,D\| = \min\{3/7, 2/7, 2/7\} = 2/7$$

$\varphi(D) = 2/7$ [Chen, *J. Number Theory*, 1991] $\geq 1/4$.

Wills Conjecture:

For any D, $$\boxed{\varphi(\mathrm{D}) \geq \frac{1}{|\mathrm{D}|+1}}$$

- Wills, Diophantine approximation, in German, 1967.
- Betke and Wills, 1972. (Confirmed for |D|=3.)
- Cusick, View obstruction problem, 1973.
- Cusick and Pomerance, 1984. (Confirmed for |D| ≤ 4.)
- Bienia et al, View obstruction and the lonely runner,

  JCT B, 1998. (New name.)
- Y.-G. Chen, On a conjecture in diophantine approximations, I – IV, *J. Number Theory*, 1990 &1991.

  (A more generalized conjecture.)

Relations

L. & Zhu, *J. Graph Theory*, 2004

Zhu, 2001

$$\chi_f(G(Z,D)) \le \chi_c(G(Z,D)) \le \frac{1}{\varphi(D)} \overset{?}{\le} |D|+1$$

Lonely Runner Conjecture

Thank You