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A Down-to-Earth Approach to Index Theory via Noncommutative Geometry

(4 Lectures)

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The focus for this short course will be the Atiyah-Singer Index Theorem for elliptic differential operators. This is one of the deepest mathematical results of the twentieth century. To give a perspective on the course, we briefly discuss this theorem now.

Since the time of Isaac Newton, differential operators have been an important tool for science, in particular, playing an important role in mathematical physics. A differential operator is a polynomial in partial derivatives, with coeffcients being functions. Three most important classes of differential operators are elliptic, parabolic, and hyperbolic. In mathematical physics, elliptic operators appear as potential equations, parabolic ones appear as heat equations, and hyperbolic ones appear as wave equations.

Among these three classes, elliptic operators have particularly nice properties. Most of the results for elliptic operators on bounded domains in Euclidean spaces also hold for elliptic operators on more general curved spaces, such as the surface of a doughnut, or of a pretzel. Elliptic operators on such closed surfaces were observed to have an especially nice property: to each such operator, an integer called the "index" of the operator can be assigned. Curiously, when you continuously perturb an elliptic operator, the index stays fixed, whereas other important numerical invariants for the operator, such as the number of independent solutions, often vary irregularly. This fact that the index doesn't change under continuous perturbations led Izrail M. Gel'fand of Moscow to conjecture that this index has a deeper meaning relating to geometry and topology. In 1963 Michael F. Atiyah and Isadore M. Singer gave an affermative answer to Gel'fand's conjecture. They showed that the index of any elliptic operator is equal to another invariant associated to the operator: a topological invariant. This result, the Atiyah-Singer Index Theorem, is at the heart of the interaction between three important branches of mathematics: analysis, topology and geometry. It gave birth to new fields of mathematical research, such as K-theory, and led to the noncommutative geometry of Alain Connes. Both Atiyah and Connes received the Fields Medal (the mathematics equivalent of the Nobel Prize) for this and related work.

In this short course historical background will be given, and the meaning and basic properties of the index will be explained. In fact the Atiyah-Singer Index Theorem holds on all sorts of very general surfaces (manifolds), but for this course we will focus on elliptic operators on Euclidean space, and prove the index theorem here (still a highly nontrivial result). The course will require background in multivariable calculus and linear algebra. An introductory survey of the additional tools (from functional analysis and topology) needed to prove the theorem will be given. Examples will be developed in two and three dimensions. The Atiyah-Singer Index Theorem opened the door to a new world of interaction between different areas of mathematics, where analytic machinery such as operator algebras can play a significant role in topology and geometry. A crystallization of this idea is noncommutative geometry, currently important particularly in today's mathematical physics. The course will be concluded by explaining some fundamental ideas and remarkable results in noncommutative geometry, applied to foliation index theorems.