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Discrete groups, von Neumann Algebras and Noncommutative Geometry

(4 Lectures)

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I am planning to give 5 lectures about various topics in the context of *Discrete groups, von Neumann algebras and non-commutative geometry.* After starting with elementary definitions, I will cover different applications of non-commutative geometry to an understanding of the structure of group rings and group C^*-algebras.

1. **The classification of von Neumann algebras**

   If first want to recall basic definitions and give various sources of examples of von Neumann algebras. Later I will sketch a first classification (due to von Neumann himself) of von Neumann algebra factors into types. A special emphasize lies on group von Neumann algebras and von Neumann algebras that appear in algebraic quantum field theory.

2. **KMS-states and Tomita-Takesaki theory**

   KMS-states appear naturally as equillibrium states in the presence of a time evolution in the algebra of observables of a physical system. We will study KMS-states from a mathematical point of view and explain the theory of Tomita-Takesaki. One consequence is that every faithful state on a von Neumann algebra is the KMS-state with respect to a naturally associated time evolution.

3. **Fredholm-modules and projections in group C^*-algebras**

   In the context of non-commutative geometry, index theory and geometry are linked through the concept of a Fredholm module. We want to explain this notion and give natural examples in the context of group algebras. We will also sketch the proof (following Connes) of non-existence of projections in the full and reduced group C^*-algebra of a free group. I will also mention Zalesski's theorem about the trace of an idempotent in a group ring.

4. **The non-zero divisor conjecture and related conjectures**

   An important conjecture about the complex group ring of a discrete torsionfree group is the absence of zero-divisors. I will indicate (following work of Linnell) how Fredholm modules can be used to prove this conjecture in the case of free groups. A related conjecture about the finiteness of group rings is also discussed and I will explain (following Elek-Szabo) how one can prove this conjecture for sofic groups.

5. **Dimension theory over finite von Neumann algebras**

   One of the advantages of finite von Neumann algebras is that they allow for a numerical dimension of modules. We introduce this dimension function and state various results about it, ranging from the theory of subfactors, ergodic theory to the theory of ℓ^2-invariants of discrete groups. In this talk I will also explain joint work with Jesse Peterson (Vanderbilt). We obtained strong information about the structure of large classes of discrete groups. The relevant invariant is the first ℓ^2-Betti number of the group.