Estimating Extremal Dependence in Time Series via the Extremogram

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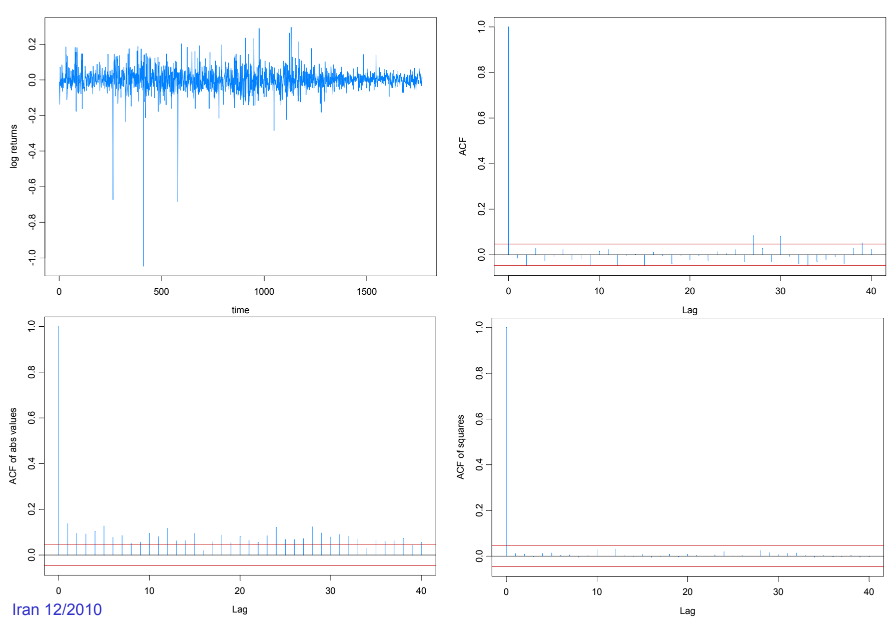
Motivating Example: Amazon-returns (May 16, 1997 – June 16, 2004



Starting point: GARCH vs SV

$X_t = \sigma_t Z_t$ (observation eqn in state-space formulation)

(i) GARCH(1,1)

$$X_t = \sigma_t Z_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 \sigma_{t-1}^2, \ \{Z_t\} \sim \text{IID}(0,1)$$

(ii) Stochastic Volatility

$$X_t = \sigma_t Z_t, \quad \log \sigma_t^2 = \phi_0 + \phi_1 \log \sigma_{t-1}^2 + \varepsilon_t, \quad \{\varepsilon_t\} \sim \text{IID N}(0,\sigma^2)$$

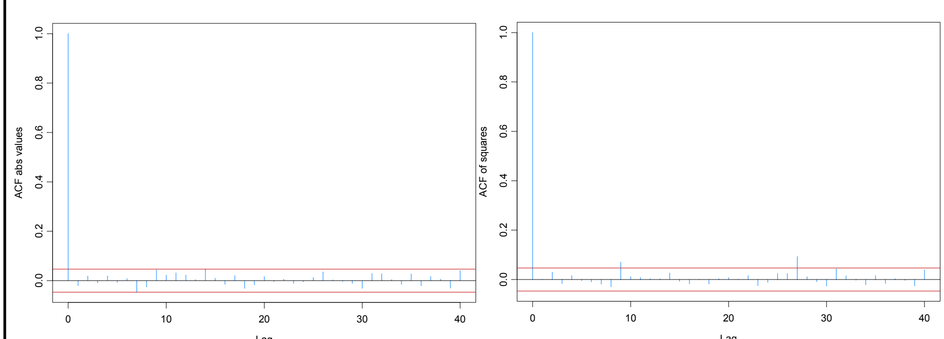
Key question:

What intrinsic (extremal?) features in the data (*if any*) can be used to discriminate between these two models?

Amazon returns (GARCH model)

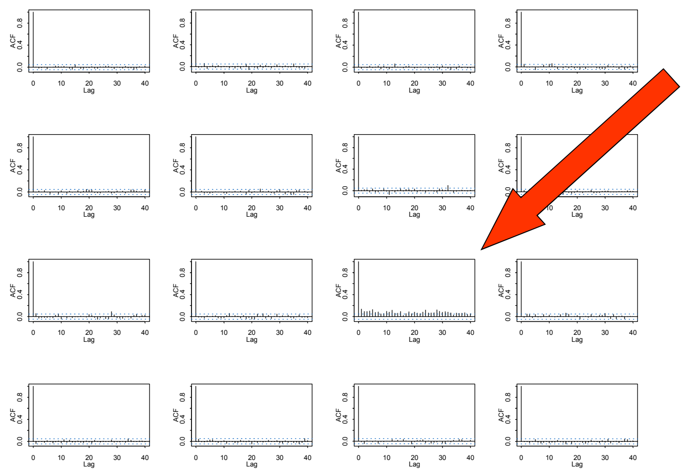
GARCH(1,1) model fit to Amazon returns:
α_0= .00002493, α_1= .0385, β_1 = .957, $X_t=(\alpha_0+\alpha_1 X^2_{t-1}+\beta_1 \sigma^2_{t-1})^{1/2}Z_t$,
$\{Z_t\}$~IID t(3.672)

Simulation from fitted GARCH(1,1) model



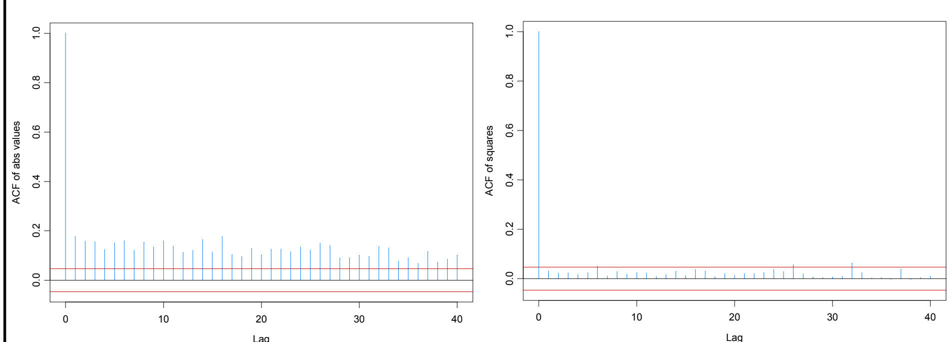
ACF Plots for Amazon

ACF of the absolute values from 15 simulated realizations from the GARCH model on previous slide.



Amazon returns (SV model)

Stochastic volatility model fit to Amazon returns: simulation based on fitted model.



Game Plan

- Extremes and time series modeling
  - A motivating example
  - Starting point: GARCH vs SV
- The Extremogram
  - Examples
  - Sufficient conditions for existence: regular variation
  - Empirical extremogram
  - Illustrations (permutation procedures)
  - Cross-extremogram (devolatilizing/deGARCHing)
- Bootstrapping the Extremogram
  - Theory & examples
- Connections with Return Times of Rare Events

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The Extremogram

The extremogram of a stationary time series $\{X_t\}$ can be viewed as the analogue of the correlogram in time series for measuring dependence in extremes (see Davis and Mikosch (2009)).

Definition: For two sets A & B ***bounded away from 0***, the ***extremogram*** is defined as

$$\rho_{A,B}(h) = \lim_{x\to\infty} P(\mathbf{X}_h \in xB \mid \mathbf{X}_0 \in xA)$$

$$= \lim_{x\to\infty} P(\mathbf{X}_0 \in xA, \mathbf{X}_h \in xB)/P(\mathbf{X}_0 \in xA),$$

for h = 0, 1, …, provided the limit exists, where $\mathbf{X}_h=(X_h,X_{h+1},\ldots,X_{h+k})$.

Remark: This definition requires that the limit exists.

a) exists for heavy-tailed time series (see forthcoming slide)
b) exists for some light-tailed time series w/ special choices of A and B.
c) extremal dependence ***depends*** on the choice of sets A & B.

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The Extremogram (cont)

If one takes $A=B=(1,\infty)$ and $k = 0$, then

$$\rho_{A,B}(h) = \lim_{x\to\infty} P(X_h > x \mid X_0 > x) = \lambda(X_0, X_h)$$

often called the ***extremal dependence coefficient*** ($\lambda = 0$ means independence or asymptotic independence).

Other choices of A and B can lead to interesting extremograms:

- $P(X_1 < -x \mid X_0 < -x)$ (negative return followed by a neg return)
- $P(X_1 > x \mid X_0 < -x)$ (neg return followed by a pos return)
- $P(X_1 + \cdots + X_4 > 2x \mid X_0 < -x)$ (neg return followed by a big pos return aggregated over next 4 days)
- $P(X_1 > x, \ldots, X_4 > x \mid X_0 > x)$ (pos return followed by a pos return in next 4 days)
- $P(\min\{X_2, X_3, X_4\} > x \mid X_0 > x, X_1 > x)$ (2 pos returns$\Rightarrow$pos return)

The Extremogram: examples

Let $A = B = (1,\infty)$, then

$$\rho_{A,B}(h) = \lim_{x\to\infty} P(X_0 > x, X_h > x)/P(X_0 > x)$$

Gaussian Processes: In this case,

$$\rho_{A,B}(h) = 0 \text{ for all } h > 0 \quad \text{(asymptotic independence).}$$

GARCH: In this case

$$\rho_{A,B}(h) > 0 \text{ for all } h > 0,$$

but decays to 0 geometrically fast.

SV process: $X_t = \sigma_t Z_t, \quad \log\sigma_t^2 = \mu + \sum_{j=0}^{\infty}\psi_j \varepsilon_{t-j}, \ \{\varepsilon_t\} \sim \text{IIDN}(0,\sigma^2)$

In this case,

$$\rho_{A,B}(h) = 0 \text{ for all } h > 0.$$

The Extremogram: examples

Let A = B = $(1,\infty)$, then

$$\rho_{A,B}(h) = \lim_{x\to\infty} P(X_0 > x, X_h > x)/P(X_0 > x)$$

AR(1): $X_t = \phi X_{t-1} + Z_t$, $\{Z_t\}$~IID Cauchy. Then

$$\rho_{A,B}(h) = \max(0, \phi^h).$$

Note if $\phi < 0$, then extremogram alternates between positive #'s and 0

MaxMA(2): Let (Z_t) be iid with Pareto distribution, i.e., $P(Z_1 > x) = x^{-\alpha}$ for $x \geq 1$, and set $X_t = \max(Z_t, Z_{t-1}, Z_{t-2})$. Then

$$\rho_{A,B}(h) = 1 \quad \text{for } h = 0.$$
$$= 2/3 \quad \text{for } h = 1$$
$$= 1/3 \quad \text{for } h = 2$$
$$= 0, \quad \text{for } h > 2.$$

Regular Variation — multivariate case

Regular variation of $\mathbf{X}=(X_1, \ldots, X_k)$: (heavy-tailed analogue of multivariate Gaussian)

(i) The radial part $|\mathbf{X}|$ is heavy-tailed, i.e.,

$$P(|\mathbf{X}|> tx)/P(|\mathbf{X}|>t) \to x^{-\alpha}.$$

(ii) The angular part $\mathbf{X} / |\mathbf{X}|$ is asymptotically independent of the radial part $|\mathbf{X}|$, i.e., there exists a random vector $\theta \in S^{k-1}$ such that

$$P(\mathbf{X}/|\mathbf{X}| \in \bullet \mid |\mathbf{X}|>t) \to_w P(\theta \in \bullet) \quad \text{as } t \to \infty.$$

($\to_w$ weak convergence on S^{k-1} = unit sphere in R^k) .

- $P(\theta \in \bullet)$ is called the spectral measure
- α is the index of $\mathbf{X}$.

Definition: A time series $\{X_t\}$ is ***regularly varying*** if all the finite dimensional distributions are regularly varying.

Regular Variation — multivariate case

RV: $P(|\mathbf{X}|> tx)/P(|\mathbf{X}|>t) \to x^{-\alpha}$ and $P(\mathbf{X}/|\mathbf{X}| \in \bullet \mid |\mathbf{X}|>t) \to_w P(\theta \in \bullet)$

Three equivalent formulations of RV:

1. Polar coordinate version:

$$P(|\mathbf{X}|> t\,x, \mathbf{X}/|\mathbf{X}| \in \bullet)/P(|\mathbf{X}|>t) \to_v x^{-\alpha} P(\theta \in \bullet)$$

2. Rectangular coordinate version:

$$\frac{P(\mathbf{X} \in t\bullet)}{P(|\mathbf{X}|>t)} \to_v \mu(\bullet)$$

μ is a measure on R^m which satisfies for $x > 0$ and A bounded away from 0,

$$\mu(xA) = x^{-\alpha}\mu(A).$$

3. Sequential version: There exists a sequence a_n such that

$$nP(a_n^{-1}\mathbf{X} \in \bullet) \to_v \mu(\bullet)$$

Regular Variation and the Extremogram

Fact: The extremogram of a RV stationary time series $\{X_t\}$ exists.

Recall that for two sets A & B bounded away from 0 (take the random vectors to be one-dimensional), the ***extremogram*** is given by

$$\rho_{A,B}(h) = \lim_{x\to\infty} P(x^{-1}X_0 \in A, x^{-1}X_h \in B)/ P(x^{-1}X_0 \in A)$$

This limit can be traced back through the limiting μ measure in defn of RV. That is, defining $\mathbf{X} = (X_0, X_1, \ldots, X_h)'$, and using

$$nP(a_n^{-1}\mathbf{X} \in \bullet) \to_v \mu(\bullet),$$

we have

$$P(a_n^{-1}(X_0,X_h) \in A\times B)/P(a_n^{-1}X_0 \in A) = P(a_n^{-1}\mathbf{X} \in A\times R^{h-1}\times B)/P(a_n^{-1}\mathbf{X} \in A\times R^h)$$

$$\to \mu(A\times R^{h-1}\times B)/\mu(A\times R^h),$$

in which case,

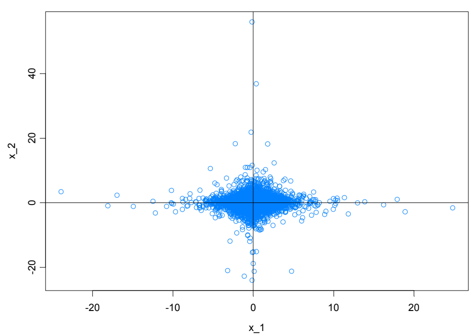
$$\rho_{A,B}(h) = \mu(A\times R^{h-1}\times B)/\mu(A\times R^h).$$

Examples of RV Time Series

Examples: 1. Let $\{X_t\}$ be iid RV($-\alpha$), then **X**$= (X_1, X_2, \ldots, X_k)$ is regularly varying with index α and *spectral distribution* that is concentrated on the axes.

Interpretation: Unlikely that X_t and X_{t+1} are very large at the same time.

Figure: plot of (X_t, X_{t+1}) for realization of length 10,000.



Extremogram:

$\rho_{A,B}(h) = 0$ for all $h > 0$.

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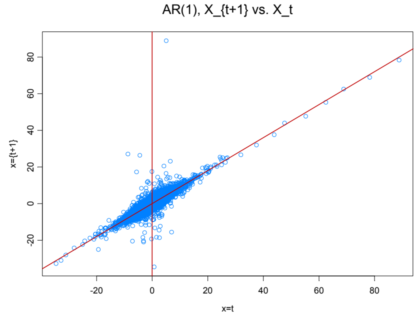
2. AR(1): $X_t = \phi X_{t-1} + Z_t$, $\{Z_t\}$~IID RV($-\alpha$).

Interpretation: If Z_t is large, then $X_t \sim Z_t$ and is independent of ϕX_{t-1}. On the other hand, $X_{t+1} \sim \phi X_t$

Figure: plot of (X_t, X_{t+1}) for realization of length 10,000 with $\phi = .9$.



Extremogram: Let A =$(1,\infty)$ and B = $(1,\infty)$, then $\rho_{A,B}(h) = \max(0, \phi^h)$.

Note if $\phi < 0$, then extremogram alternates between positive #'s and 0.

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Examples of RV Time Series

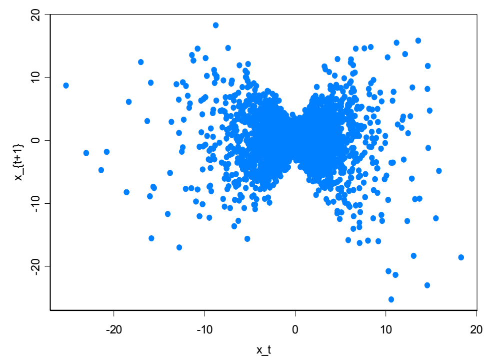
3. GARCH(1,1): $X_t=(\alpha_0+\alpha_1 X^2_{t-1} +\beta_1\sigma^2_{t-1})^{1/2}Z_t$, $\{Z_t\}$~IID.
$\alpha_0=1$, $\alpha_1=1$, $\beta_1=0$

It turns out that finite dim'l distrs are regularly varying (see Mikosch and Stăriciă (2000))

Figure: plot of (X_t, X_{t+1}) for realization of length 10,000.

Extremogram:

$\rho_{A,B}(h) > 0$ for all h.



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Examples of RV Time Series

4. SV model $X_t = \sigma_t Z_t$

Suppose $Z_t \sim RV(-\alpha)$ and

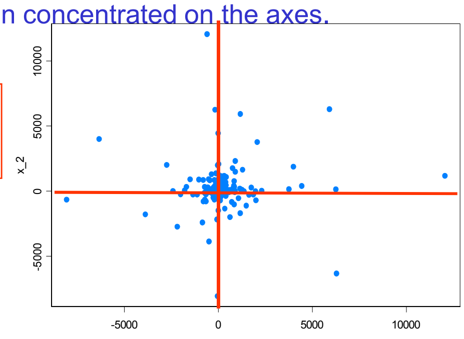
$$\log\sigma_t^2 = \sum_{j=-\infty}^{\infty}\psi_j\varepsilon_{t-j},\ \sum_{j=-\infty}^{\infty}\psi_j^2 < \infty, \{\varepsilon_t\} \sim \mathrm{IID\,N}(0,\sigma^2).$$

Then $\mathbf{Z}_n=(Z_1,\ldots,Z_n)'$ is regulary varying with index α and so is

$$\mathbf{X}_n= (X_1,\ldots,X_n)' = \mathrm{diag}(\sigma_1,\ldots,\sigma_n)\,\mathbf{Z}_n$$

with spectral distribution concentrated on the axes.

Figure: plot of (X_t,X_{t+1}) for realization of 10,000.



Extremogram:

$\rho_{A,B}(h) = 0$ for all $h > 0$.

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The Empirical Extremogram

A natural estimator of the extremogram is the empirical extremogram defined by

$$\hat{\rho}_{A,B}(h) = \frac{\frac{m}{n}\sum_{t=1}^{n-h} I_{\{a_m^{-1}X_t \in A, a_m^{-1}X_{t+h} \in B\}}}{\frac{m}{n}\sum_{t=1}^{n} I_{\{a_m^{-1}X_t \in A\}}},$$

where $m \to \infty$ with $m/n \to 0$ and a_m is the 1-m/n quantile of $|X_t|$. Note that the limit of the expectation of the numerator is

$$mP\,(a_m^{-1}X_0 \in A,\ a_m^{-1}X_h \in B) \to \mu(A \times R^{h-1} \times B),$$

where μ is the measure defined in the statement of regular variation of the vector $\mathbf{X} = (X_0, X_1, \ldots, X_h)'$. Hence the empirical estimate is asymptotically "unbiased". Under suitable mixing conditions, a CLT for the empirical estimate is established in D&M (2009).

The Empirical Extremogram — central limit theorem

$$\hat{\rho}_{A,B}(h) = \frac{\frac{m}{n}\sum_{t=1}^{n-h} I_{\{a_m^{-1}X_t \in A, a_m^{-1}X_{t+h} \in B\}}}{\frac{m}{n}\sum_{t=1}^{n} I_{\{a_m^{-1}X_t \in A\}}}$$

After first establishing a joint CLT for the numerator and denominator, we obtain the limit result

$$(n/m)^{1/2}(\hat{\rho}_{A,B}(h) - \rho_m(h)) \to_d N(0, \sigma^2(A,B)),$$

where $\rho_m(h)$ is the ratio of expectations (***pre-asymptotic bias***),

$$P\,(a_m^{-1}X_0 \in A,\ a_m^{-1}X_h \in B) / P\,(a_m^{-1}X_0 \in A).$$

Now provided a bias condition, such as

$$(n/m)^{1/2}\left(mP\,(a_m^{-1}X_0 \in A,\ a_m^{-1}X_h \in B) - \mu_h(A \times B)\right) \to 0,$$

holds, then $\rho_m(h)$ can be replaced with $\rho_{A,B}(h)$. This condition can often be difficult to check.

Spectral Analysis for the Extremogram

For a fixed *nice* set C, define $\tau_h(C) = \gamma_{CC}(|h|)$ and $\tau_0(C) = \mu(C)$, i.e.,

$$nP(a_n^{-1}X_0 \in C, a_n^{-1}X_h \in C) \to \tau_h(C)$$

The spectral density is then defined by

$$f(\lambda) = \tau_0(C) + 2\sum_{h=1}^{\infty}\cos(\lambda h)\tau_h(C), \quad \lambda \in [0,\pi].$$

The sample version of the spectral density is give by the periodogram

$$I_n(\lambda) = \hat{\gamma}_n(0) + 2\sum_{h=1}^{n-1}\cos(\lambda h)\hat{\gamma}_n(h), \quad \lambda \in [0,\pi],$$

where $\hat{\gamma}_n(h) = \frac{m_n}{n}\sum_{t=1}^{n-h}(I_{\{a_m^{-1}X_t \in C\}} - P(a_m^{-1}X_t \in C)), \quad h \geq 0.$

In the standard time series setting, the periodogram estimator is not consistent for f(λ). Instead, a lag-window estimator is used.

Spectral Analysis for the Extremogram

A lag-window estimator for f is defined by

$$\hat{f}_n(\lambda) = \hat{\gamma}_n(0) + 2\sum_{h=1}^{r_n}\cos(\lambda h)\hat{\gamma}_n(h), \quad \lambda \in [0,\pi],$$

where $r_n \to \infty$ and $m/r_n \to 0$. This estimator is asymptotically unbiased and consistent for

$$f(\lambda) = \tau_0(C) + 2\sum_{h=1}^{\infty}\cos(\lambda h)\tau_h(C).$$

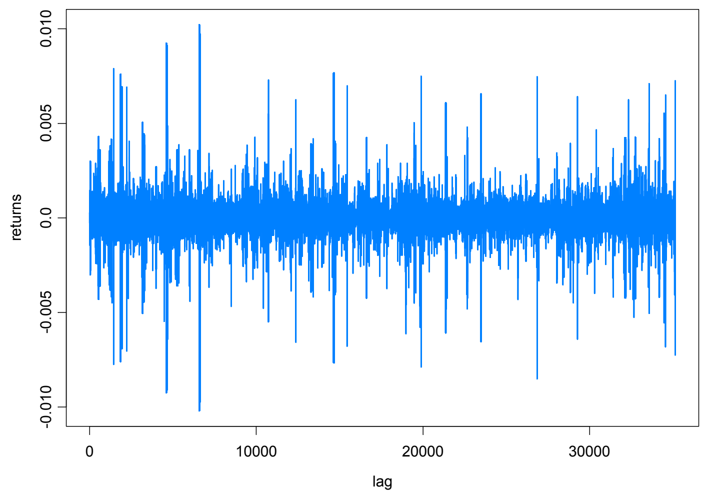
Theorem. Under our mixing condition and general setup,

$$\lim_{n\to\infty} EI_n(\lambda) = \lim_{n\to\infty} E\hat{f}_n(\lambda) = f(\lambda), \quad \lambda \in (0,\pi).$$

If, in addition, $m_n r_n^2 = O(n)$, then

$$\lim_{n\to\infty} E(\hat{f}_n(\lambda) - f(\lambda))^2 = 0, \quad \lambda \in (0,\pi).$$

Application to Five-Minute Return Data (US/DM) exchange



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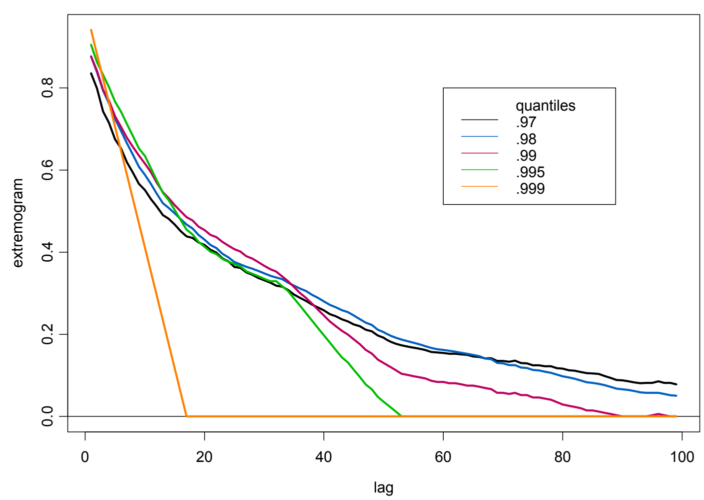
Application to Five-Minute Return Data (US/DM) exchange

Extremogram absolute values: choice of threshold a_m



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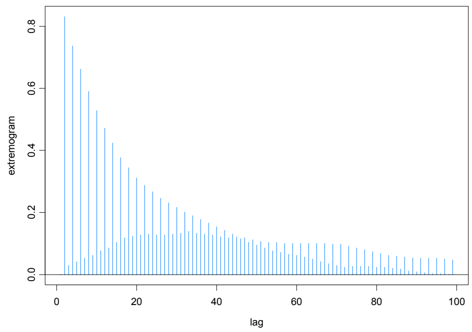
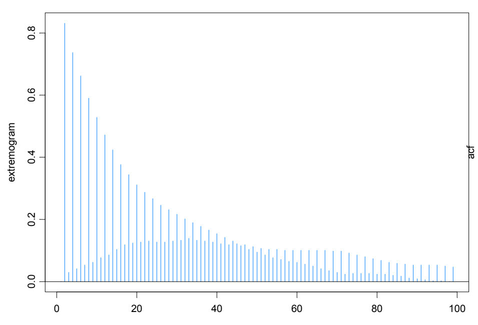
Application to Five-Minute Return Data (US/DM) exchange

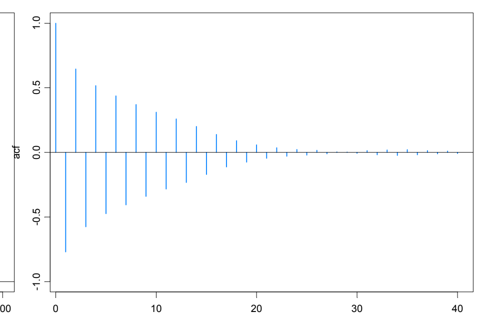
Extremogram A=B=(1,∞)



Application to Five-Minute Return Data (US/DM) exchange

Extremogram A=B=(1,∞)

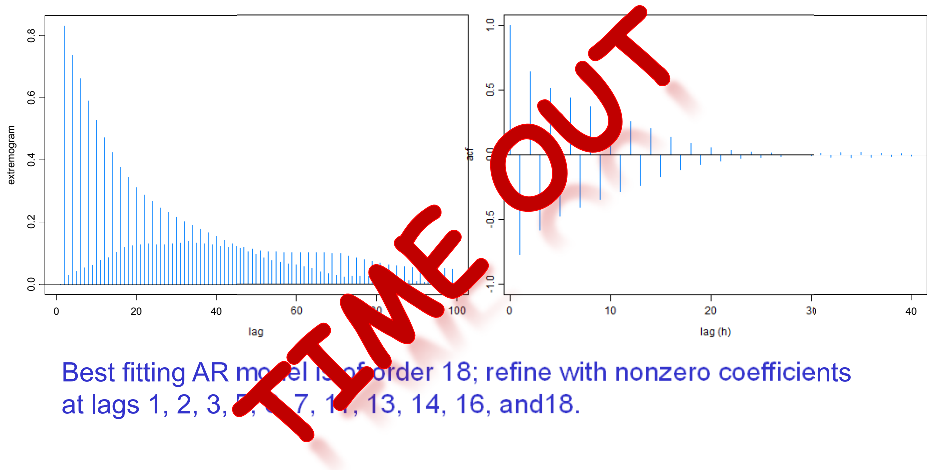




Best fitting AR model is of order 18; refine with nonzero coefficients at lags 1, 2, 3, 5, 6, 7, 11, 13, 14, 16, and18.

Application to Five-Minute Return Data (US/DM) exchange

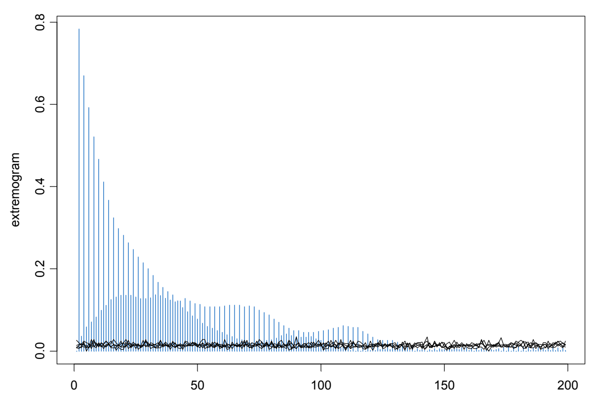
Extremogram A=B=(1,∞)



Best fitting AR model is of order 18; refine with nonzero coefficients at lags 1, 2, 3, 5, 6, 7, 11, 13, 14, 16, and18.

Time out: Resampling and Testing for Serial Dependence

A natural way (*not often used in time series*) for testing serial correlation is to compute the ACF for random permutations of the data. If the sample ACF appears more ***extreme*** than the ACFs based on random permutations, then there is evidence of serial correlation. We apply the same idea to the extremogram.



Time out: Resampling and Testing for Serial Dependence

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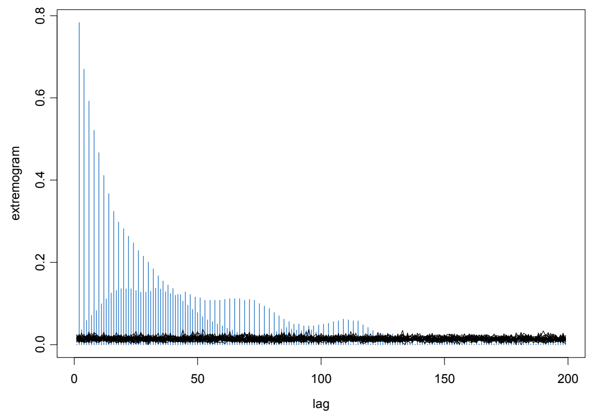
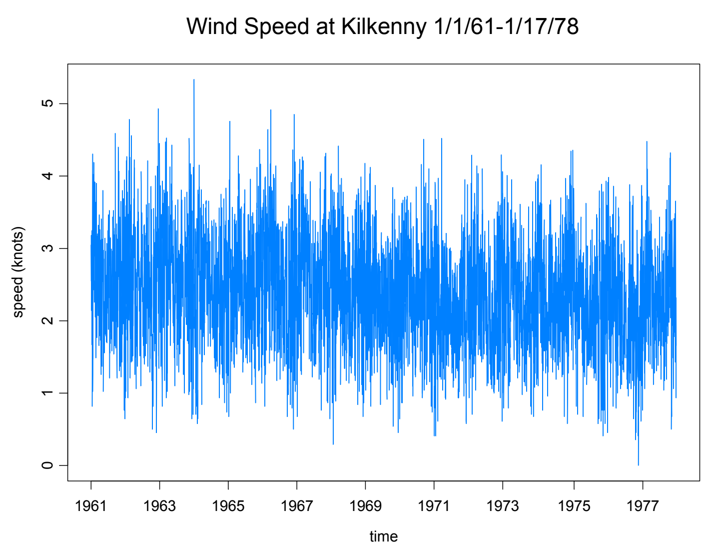
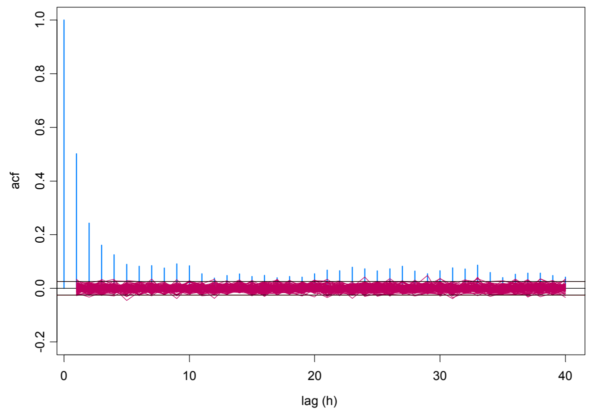


Time out: Illustration with ACF (Windspeed at Kilkenny)



Time out: Illustration with ACF

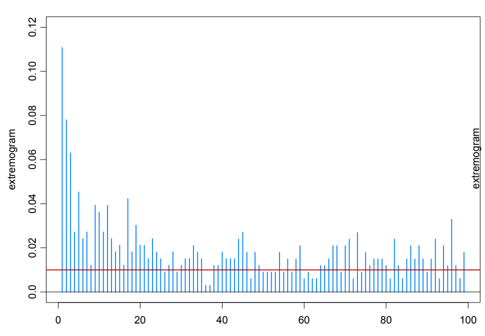
In plotting the sample ACF, one normally includes the $\pm 1.96/\text{sqrt}(n)$ bounds (95% CI under the assumption of iid noise). One could use the permutation idea here as well.



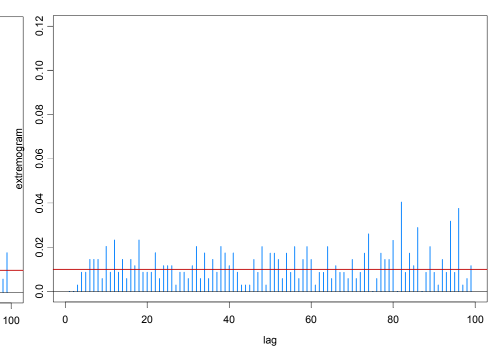
Application to Five-Minute Return Data (US/DM) exchange

Extremogram for residuals from subset AR(18) and from GARCH
$A=B=(1,\infty)$

Residuals from AR



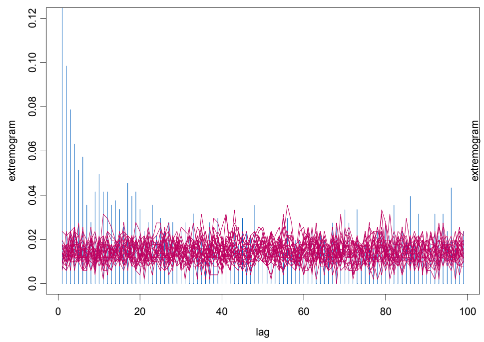
Residuals from GARCH



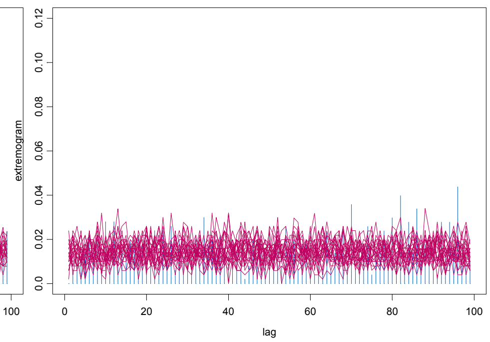
Application to Five-Minute Return Data (US/DM) exchange

Extremogram for residuals from subset AR(18) and from GARCH
$A=B=(1,\infty)$

Residuals from AR

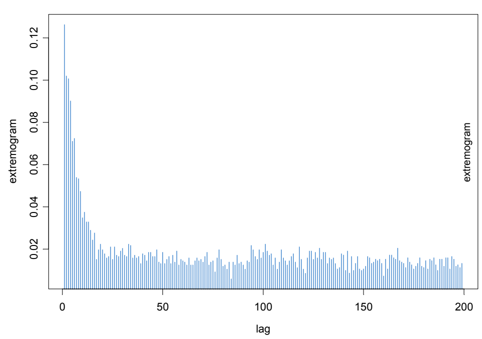


Residuals from GARCH

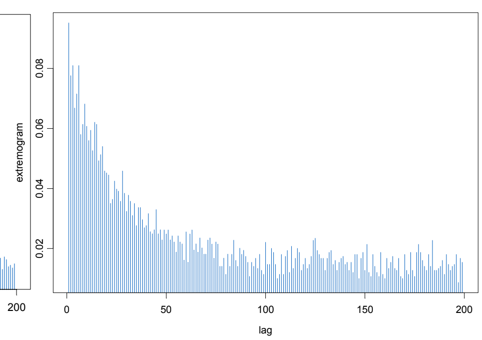


Extremogram of a SV Process

SV Process: $X_t=\sigma_t Z_t$, $\{Z_t\} \sim$ IID t_4 ; $\log \sigma_t = .9 \log \sigma_{t-1} + \varepsilon_t$
GARCH(1,1): $X_t=(.1+.14 X^2_{t-1} +.83\sigma^2_{t-1})^{1/2} Z_t$, $\{Z_t\} \sim$ IID N(0,1),



SV



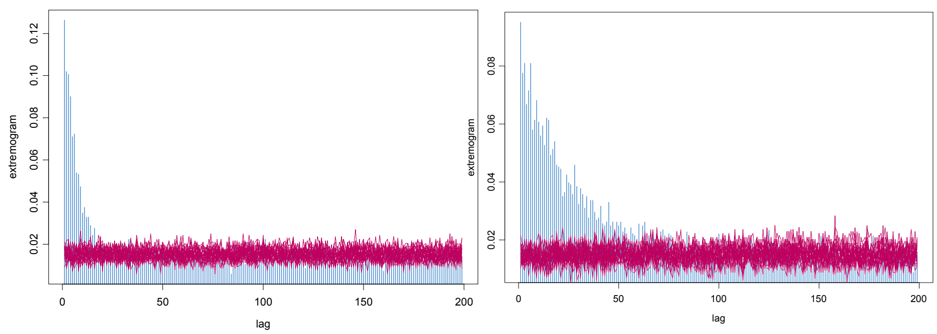
GARCH

Threshold = .97 quantile

Extremogram of a SV Process

SV Process: $X_t=\sigma_t Z_t$, $\{Z_t\} \sim$ IID t_4 ; $\log \sigma_t = .9 \log \sigma_{t-1} + \varepsilon_t$

GARCH(1,1): $X_t=(.1+.14 X^2_{t-1} +.83\sigma^2_{t-1})^{1/2}Z_t$, $\{Z_t\} \sim$ IID N(0,1),



SV GARCH

Threshold = .97 quantile

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Extremogram of a Max-MA(2)

Example: Let (Z_t) be iid with Pareto distribution, i.e., $P(Z_1 > x) = x^{-\alpha}$ for $x \geq 1$, and set $X_t = \max(Z_t, Z_{t-1}, Z_{t-2})$. Then

$$nP(X_1 > xn^{1/\alpha}) \to 3x^{-\alpha} \text{ and } F^n(xn^{1/\alpha}) \to \exp(-3x^{-\alpha}).$$

On the other hand,

$$P(n^{-1/\alpha} M_n \leq x) = P(n^{-1/\alpha} \max(Z_{-1}, \ldots, Z_n) \leq x) \to \exp(-x^{-\alpha}) = \exp(-1/3\ 3x^{-\alpha}),$$

which implies that the extremal index is $\theta = 1/3$.

The extremogram with $A = B = (1, \infty)$ is

$$\lim_n P(X_h > n^{1/\alpha} \mid X_0 > n^{1/\alpha}) = 1 \quad \text{for } h = 0.$$
$$= 2/3 \quad \text{for } h = 1$$
$$= 1/3 \quad \text{for } h = 2$$
$$= 0, \quad \text{for } h > 2.$$

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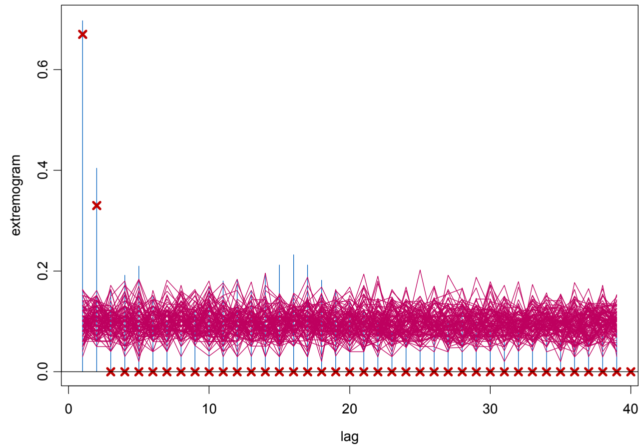
Extremogram of a Max-MA(2)

Extremogram: $\lim_n P(X_h > n^{1/\alpha} \mid X_0 > n^{1/\alpha}) = 2/3, 1/3, 0$ for $h = 1$, $h=2$, and for $h > 3$, respectively. Blue = sample



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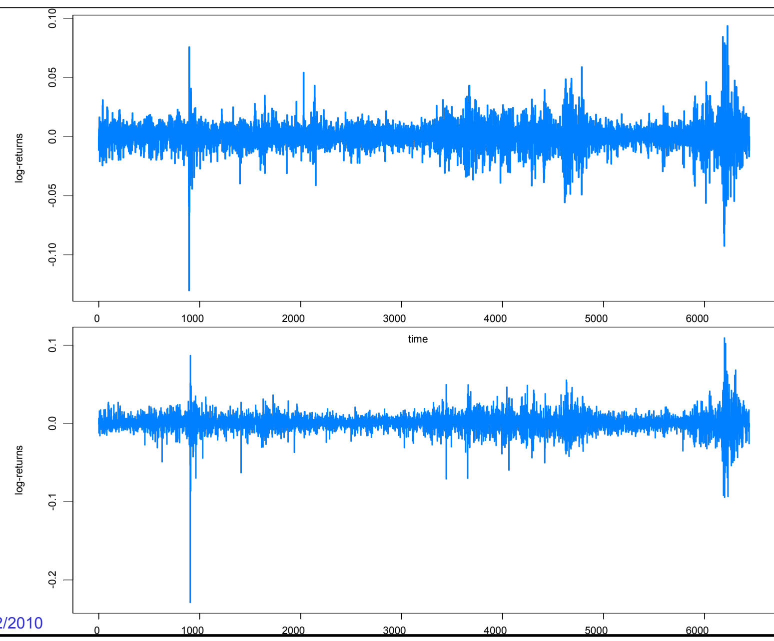
Log-returns for FTSE and S&P (Apr 4, `84-Oct 2, `09



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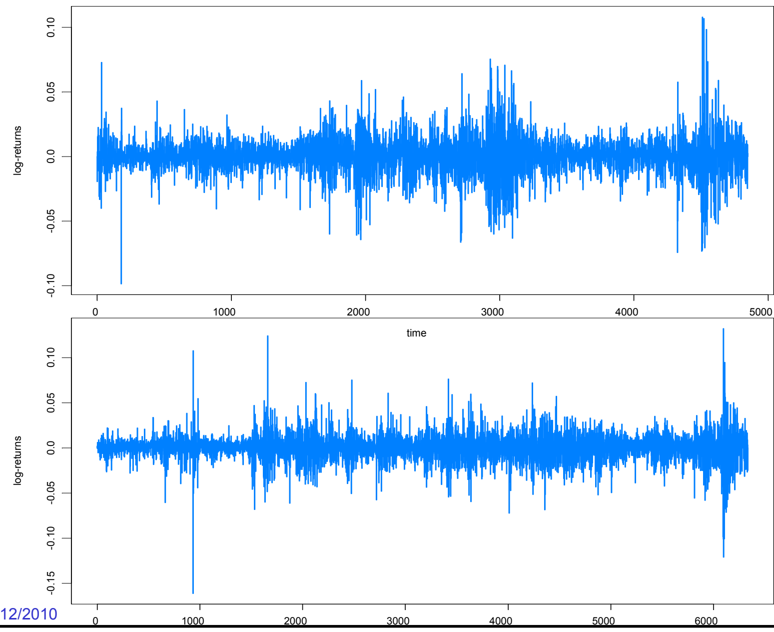
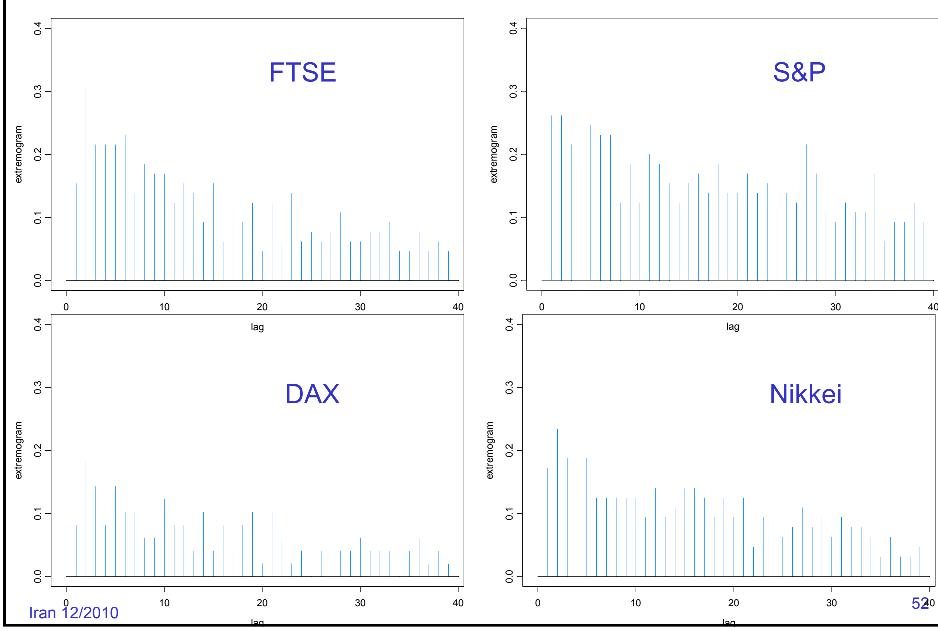
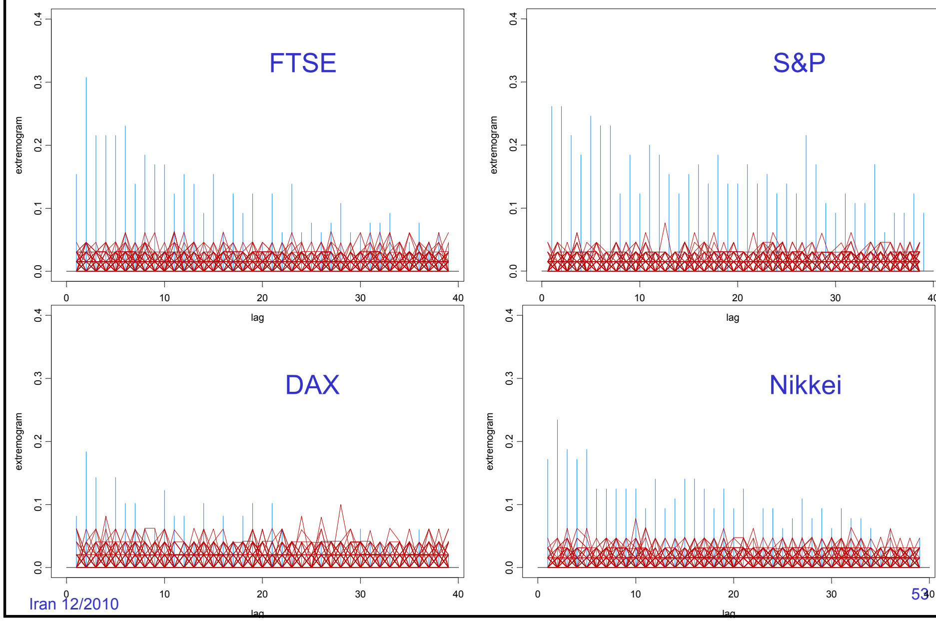
Log-returns for DAX and Nikkei (Apr 4, `84-Oct 2, `09



Extremogram for FTSE, S&P, DAX, Nikkei



Extremogram for FTSE, S&P, DAX, Nikkei



Cross-Extremogram

The cross-extremogram measures extremal dependence between two or more series. Suppose we have two time series $\{X_t\}$ and $\{Y_t\}$

Definition: For two sets A & B ***bounded away from 0***, the ***cross-extremogram*** is defined as

$$\rho_{A,B}(h) = \lim_{x\to\infty} P(Y_h \in xB \mid X_0 \in xA)$$

For example, if X_t and Y_t represent log-returns of two stocks, then one might be interested in extremal dependence of negative returns. It may seem natural to take $A=B=(-\infty,-1]$, so that

$$\rho_{A,B}(h) = \lim_{x\to\infty} P(Y_h < -x \mid X_0 < -x).$$

Cross-Extremogram

As before, we estimate

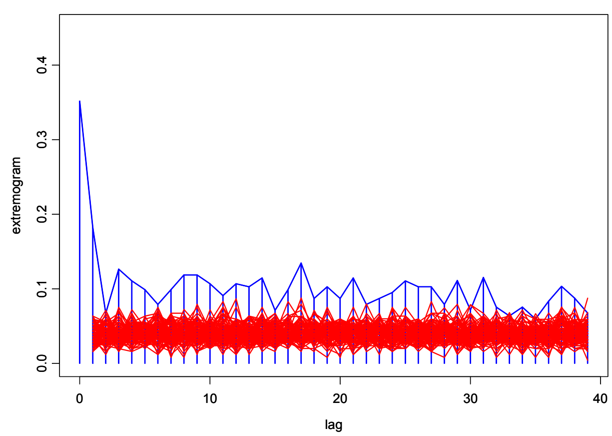
$$\rho_{A,B}(h) = \lim_{x\to\infty} P(Y_h \in xB \mid X_0 \in xA)$$

by

$$\hat{\rho}_{A,B}(h) = \frac{\frac{m}{n}\sum_{t=1}^{n-h} I_{\{a_{m,1}^{-1}X_t \in A,\, a_{m,2}^{-1}Y_{t+h} \in B\}}}{\frac{m}{n}\sum_{t=1}^{n} I_{\{a_{m,1}^{-1}X_t \in A\}}}$$

Problem: For log-returns, heteroskedasticity can produce ***spurious*** extremograms. That is, volatility in both series (which tends to happen in unison) produces large extremograms.

Cross-Extremogram FTSE and SP



Cross-Extremogram

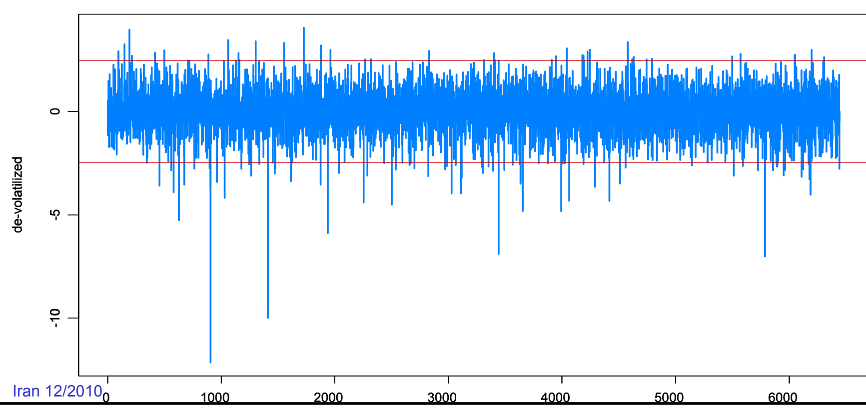
Strategy: Devolatilize the component series before computing the extremogram. This is ***analogous*** to the issue of spurious cross-correlations in a time series without whitening the series first.

Devolatilizing (deGARCHing) S&P

Extremogram for S&P: significant for large number of lags ~40+

Devolatilize S&P by fitting GARCH(1,1):

$$X_t=(6.28e{-}7+.057\,X^2_{t-1}+.939\sigma^2_{t-1})^{1/2}Z_t, \quad \{Z_t\} \sim \text{IID } t(6.14),$$

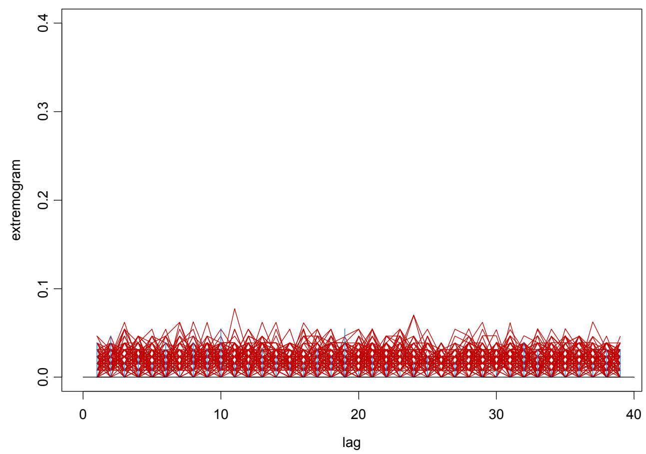


Devolatilizing S&P

Extremogram for S&P: significant for large number of lags ~40+

Devolatilize S&P by fitting GARCH(1,1):

$X_t=(6.28e\text{-}7+.057\ X^2_{t-1}+.939\sigma^2_{t-1})^{1/2}Z_t$, $\{Z_t\}$ ~ IID t(6.14),

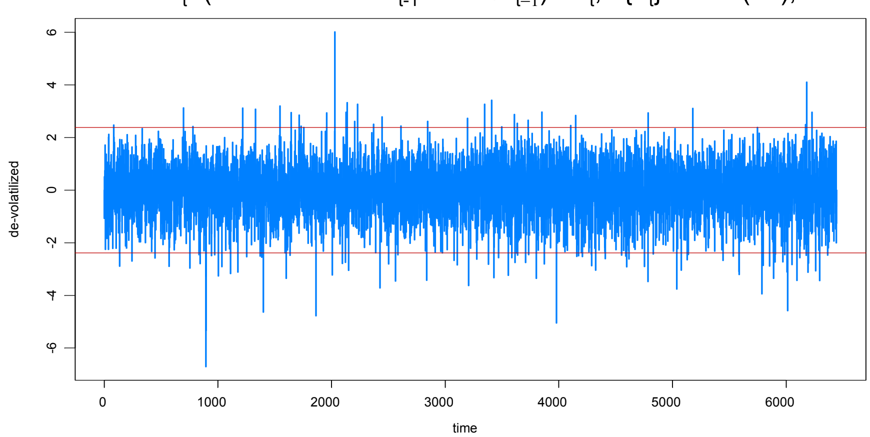


Devolatilizing (deGARCHing) FTSE

Extremogram for FTSE: significant for large number of lags ~40+

Devolatilize FTSE by fitting GARCH(1,1):

$X_t=(1.32e\text{–}6+.084\ X^2_{t-1}+.904\sigma^2_{t-1})^{1/2}Z_t$, $\{Z_t\}$ ~ IID t(13),

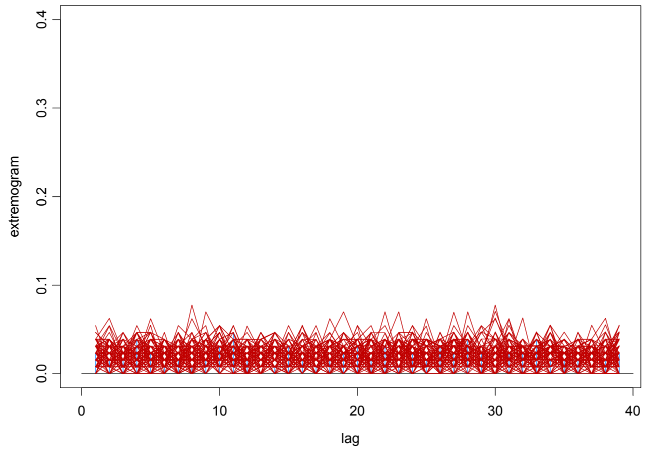


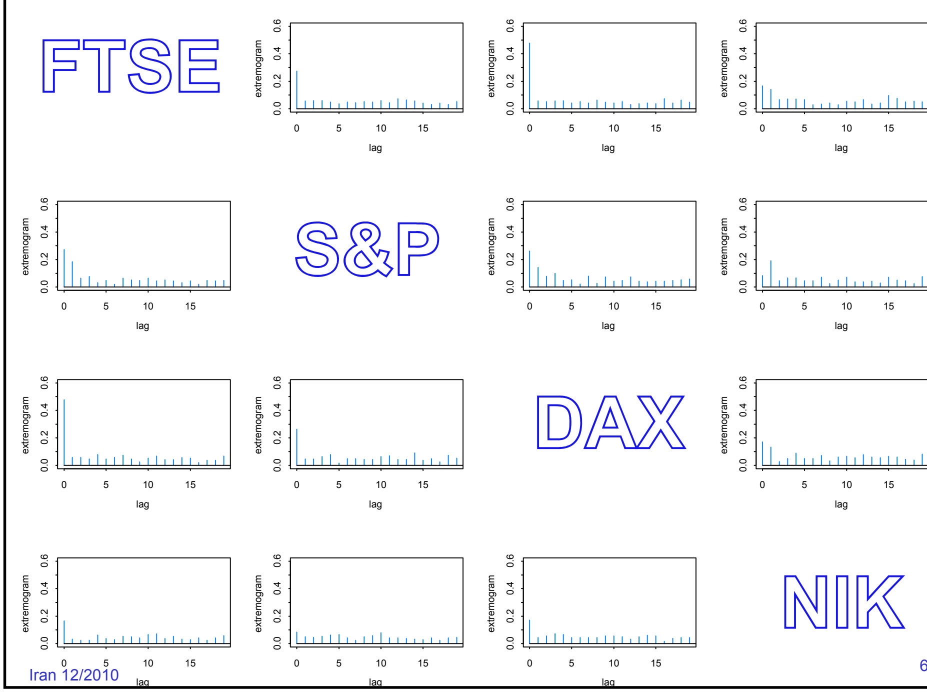
Devolatilizing FTSE

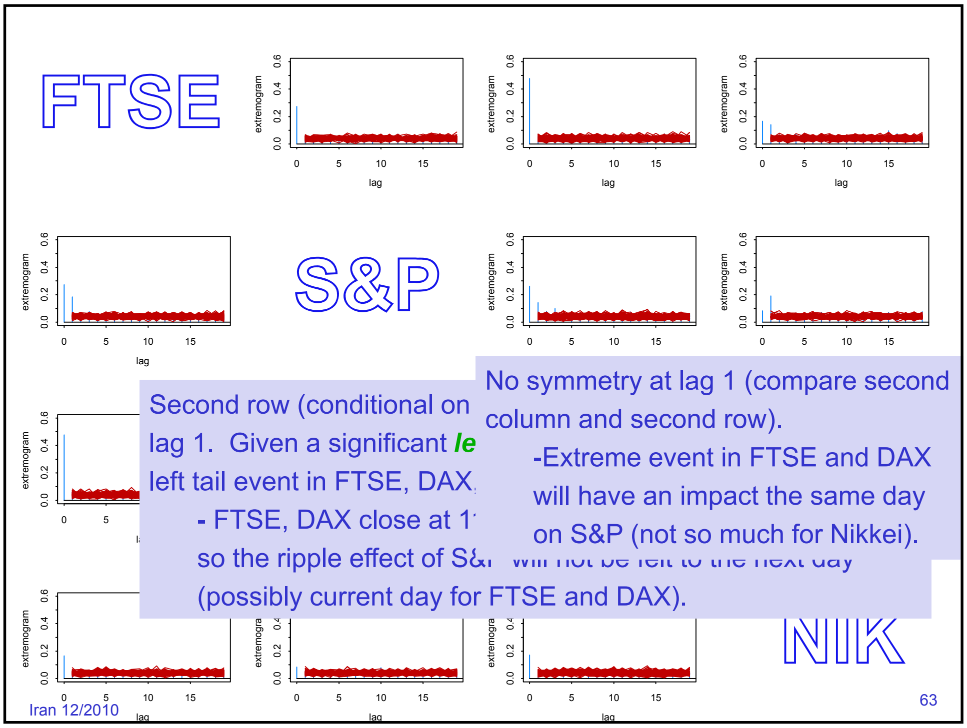
Extremogram for FTSE: significant for large number of lags ~40+

Devolatilize FTSE by fitting GARCH(1,1):

$X_t=(6.28e\text{-}7+.057\, X^2_{t-1}+.939\sigma^2_{t-1})^{1/2}Z_t$, $\{Z_t\} \sim$ IID t(6.14),







Bootstrapping the Extremogram

The stationary bootstrap, introduced by Politis and Romano (1994) seems well suited for the extremogram.

Stationary Bootstrap Setup: Have data $X_1, \ldots, X_n$ and construct BS sample as follows:

- $K_1, K_2, \ldots,$ be iid uniform on {1,…, n}
- $L_1, L_2, \ldots,$ be iid geometric(p_n)

The BS sample $X_1^*, \ldots, X_n^*$ is given by the first n observations in the sequence.

$$X_{K_1}, \ldots, X_{K_1+L_1-1}, X_{K_2}, \ldots, X_{K_2+L_2-1}, \ldots, X_{K_N}, \ldots, X_{K_N+L_N-1}$$

where

$$N = \inf\{i \geq 1 : L_1 + \cdots + L_i \geq n\}.$$

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Bootstrapping the Extremogram

$$X_{K_1},\ldots,X_{K_1+L_1-1},X_{K_2},\ldots,X_{K_2+L_2-1},\ldots,X_{K_N},\ldots,X_{K_N+L_N-1}$$

- $K_1, K_2, \ldots,$ be iid uniform on $\{1,\ldots, n\}$
- $L_1, L_2, \ldots,$ be iid geometric(p_n)

Remarks

- Procedure is similar to the block bootstrap method
- Each block has a random length given by independent geometrics, $L_1, L_2, \ldots$.
- Mean block size is $1/p_n$
- Mean number of blocks is np_n
- By the previous two bullet points, we require

$$p_n \to 0,\ np_n \to \infty.$$

Bootstrapping the Extremogram (cont)

The extremogram, computed from either the sample or BS sample, are ratios of partial sums of the form,

$$\hat{P}_n(C) = \frac{m_n}{n}\sum_{t=1}^{n} I_{\{a_m^{-1}X_t \in C\}} \quad \text{and} \quad \hat{P}_n^*(C) = \frac{m_n}{n}\sum_{t=1}^{n} I_{\{a_m^{-1}X_t^* \in C\}}.$$

Theorem . Assuming our general setup (mixing conditions + regular variation, etc), and the growth conditions,

$$np_n \to \infty, \quad np^2/m_n \to \infty,$$

we have $E^*\hat{P}_n^*(C) \xrightarrow{P} \mu(C)$ and $ms_n^2 = \mathrm{Var}^*((n/m)^{1/2}\hat{P}_n^*(C)) \xrightarrow{P} \sigma^2(C).$

Moreover,

$$\sup_x |P((n/m)^{1/2}(ms_n^2)^{-1/2}(\hat{P}_n^*(C)-\hat{P}_n(C)) \le x \mid X_1,\ldots,X_n) - \Phi(x)| \xrightarrow{P} 0$$

Bootstrapping the Extremogram (cont)

The sample extremogram and its BS counterpart are:

$$\hat{\rho}_{A,B}(h) = \frac{\frac{m}{n}\sum_{t=1}^{n-h} I_{\{a_m^{-1}X_t\in A, a_m^{-1}X_{t+h}\in B\}}}{\frac{m}{n}\sum_{t=1}^{n} I_{\{a_m^{-1}X_t\in A\}}} \qquad \hat{\rho}^*_{A,B}(h) = \frac{\frac{m}{n}\sum_{t=1}^{n-h} I_{\{a_m^{-1}X^*_t\in A, a_m^{-1}X^*_{t+h}\in B\}}}{\frac{m}{n}\sum_{t=1}^{n} I_{\{a_m^{-1}X^*_t\in A\}}}$$

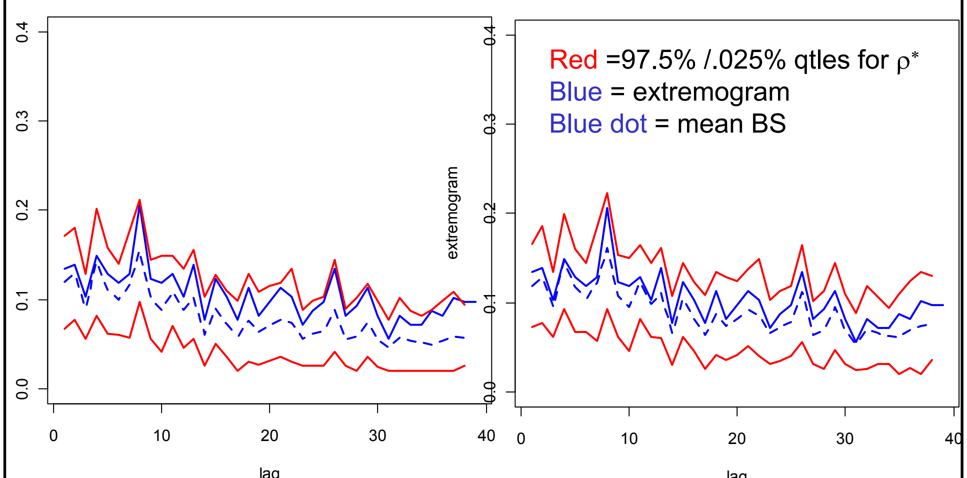
Theorem . Assuming our general setup (mixing conditions + regular variation, etc), and the growth conditions,

$$np_n \to \infty, \quad np^2/m_n \to \infty,$$

we have

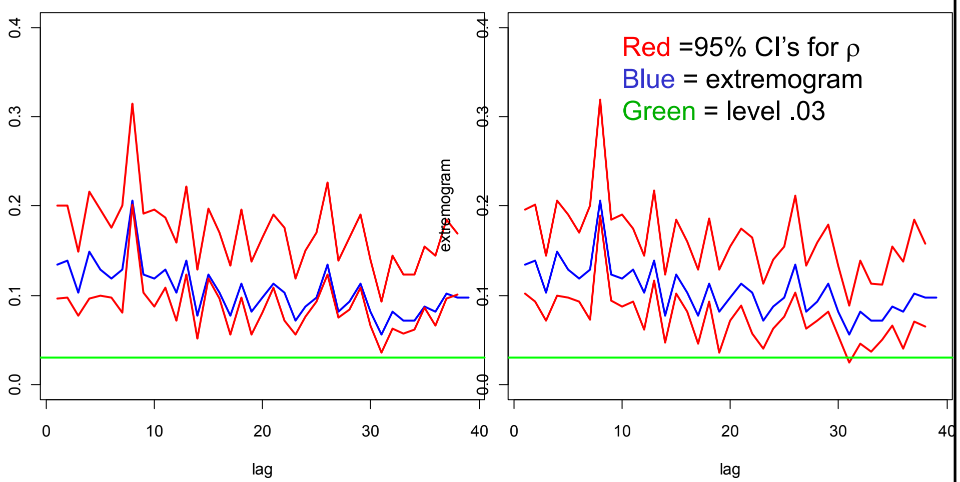
$$\sup_x | P((n/m)^{1/2}(\hat{\rho}^*_{A,B}(h) - \hat{\rho}_{A,B}(h)) \le x \mid X_1,\ldots,X_n) -$$
$$P((n/m)^{1/2}(\hat{\rho}_{A,B}(h) - \rho_m(h)) \le x) | \xrightarrow{P} 0$$

Application to FTSE (lower tail)



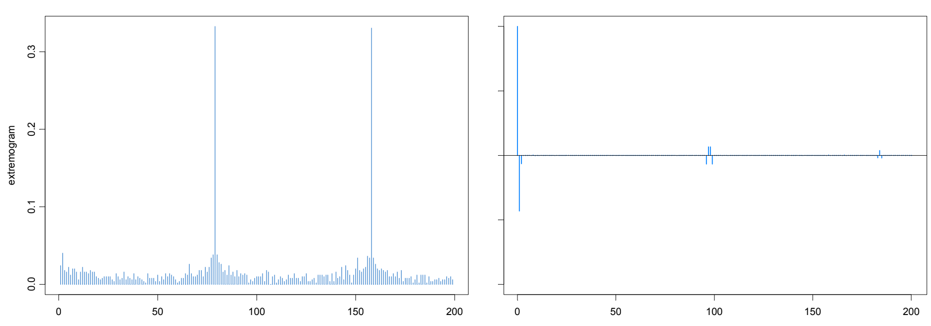
p_n =1/50 $\qquad$ p_n =1/200

Application to FTSE (lower tail)



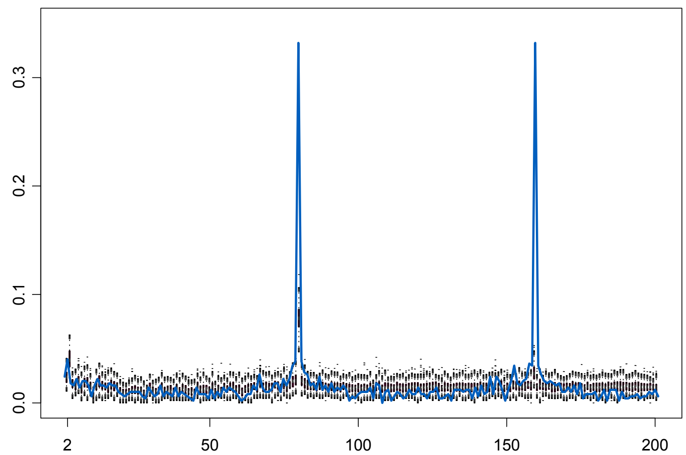
p_n =1/50 p_n =1/200

Application to 5-Minute Return Data (Goldman Sachs)



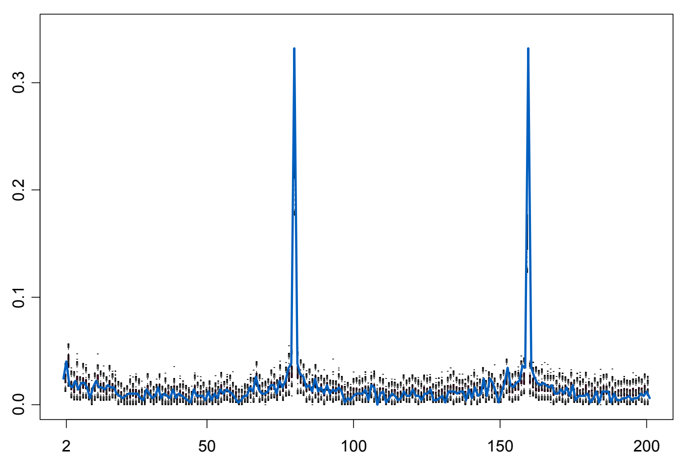
Significant extremogram at lags 78 and 156 ~ 6 hrs 30 mins & 13hrs

Bootstrap Application to 5 min Goldman-Sachs



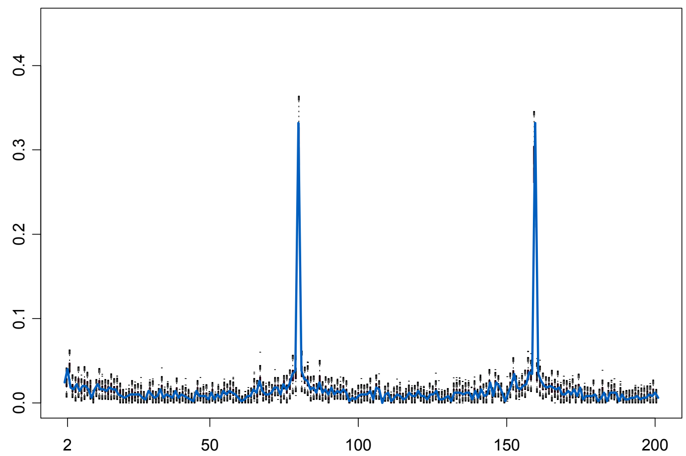
p_n =.02 (mean block size is 50)
BS reps = 100

Bootstrap Application to 5 min Goldman-Sachs



p_n =.005 (mean block size is 200)
BS reps = 100

Bootstrap Application to 5 min Goldman-Sachs



p_n =.001 (mean block size is 1000)
BS reps = 100

Connections with Return Times (of rare events)

This is an idea due to ***Geman and Chang (2009)***:

Setup:

- $\{X_t\}$ time series—think log-returns, for example.
- ξ_v, ξ_{1-v} are the vth and (1-v)th quantile of the of the marginal distribution.

Define the exceedance (or stopping times) times τ_j by

$$\tau_1 = \min\{ t \geq 1: X_t < \xi_v \text{ or } X_t < \xi_{1-v}\}$$

$$\tau_{j+1} = \min\{ t \geq \tau_j: X_t < \xi_v \text{ or } X_t < \xi_{1-v}\} , j \geq 0.$$

The inter-arrival (or return times) are

$$T_j = \tau_j - \tau_{j-1} \quad j \geq 1.$$

These are the times between occurrences of rare events (number of tosses of a coin until next head).

Connections with Return Times (of rare events)

For ***nice*** time series, like iid observations, the T_j's are iid with a geometric distribution,

$$P(T_j = k) = (1-p)^{k-1}p, \quad k=1,2, \ldots,$$

$$p = P(X_t < \xi_v \text{ or } X_t > \xi_{1-v}) = 2v.$$

Recall for a geometric rv,

$$E(T_1) = 1/p.$$

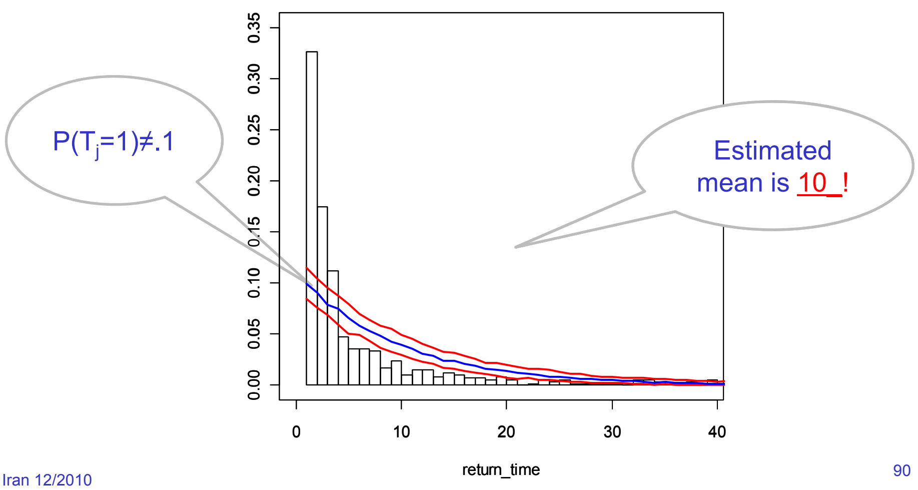
Note: This is the ***backstory*** behind the term 100 year flood, or 100 year *blank*, which corresponds to the threshold x such that the expected time until x is exceeded is 100. (In this case, p = .01, x= $\xi_{.99}$.)

Idea: For v fixed (can do one-sided tail), look at the histogram of return times and compare against a geometric distribution.

Connections with Return Times (of rare events)

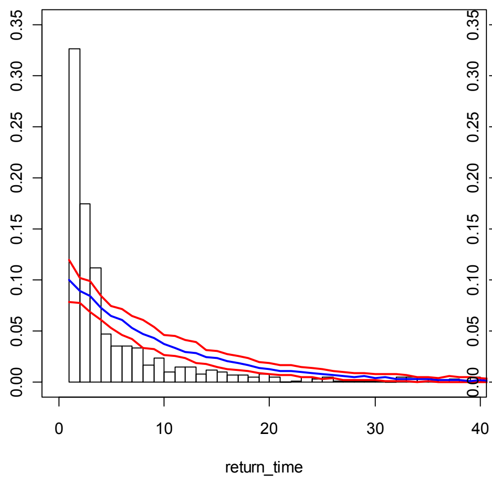
Idea: For v fixed (can do one sided tail), look at the histogram of return times and compare against a geometric distribution.

Example with BAC, v=.05 ⇒ geometric(p=.1)

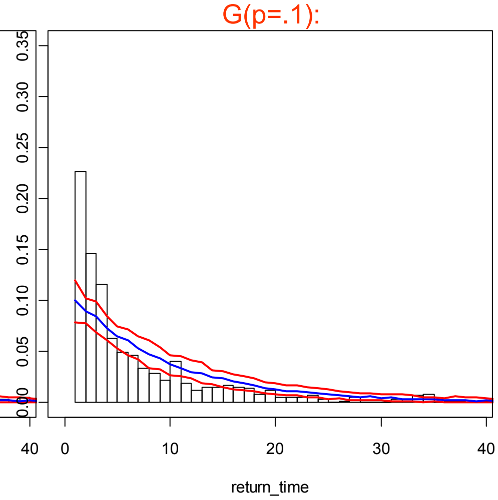


Connections with Return Times (Daily Returns for DAC)

BAC, 2 tail, v=.05 ⇒ G(p=.1)



BAC, lower only v=.01 ⇒ G(p=.1):



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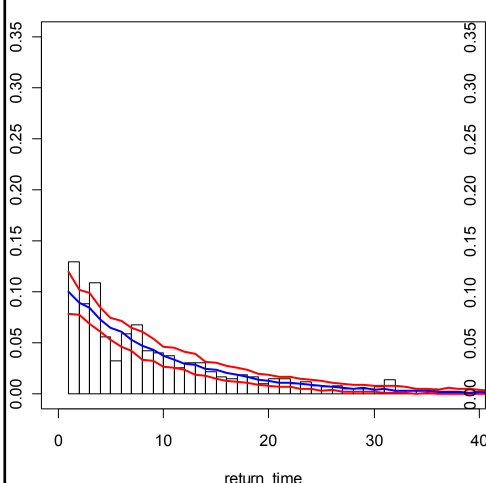
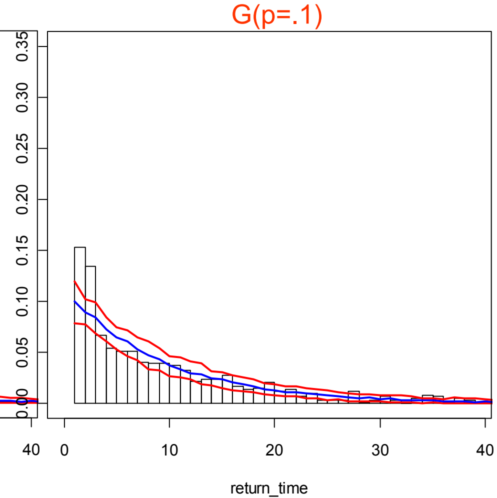
Connections with Return Times

BAC devolatilized
v=.05 ⇒ G(p=.1)



BAC devolatilized
lower tail only v=.01 ⇒ G(p=.1)



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Connections with Return Times (of rare events)

Question: What is the connection with the extremogram?

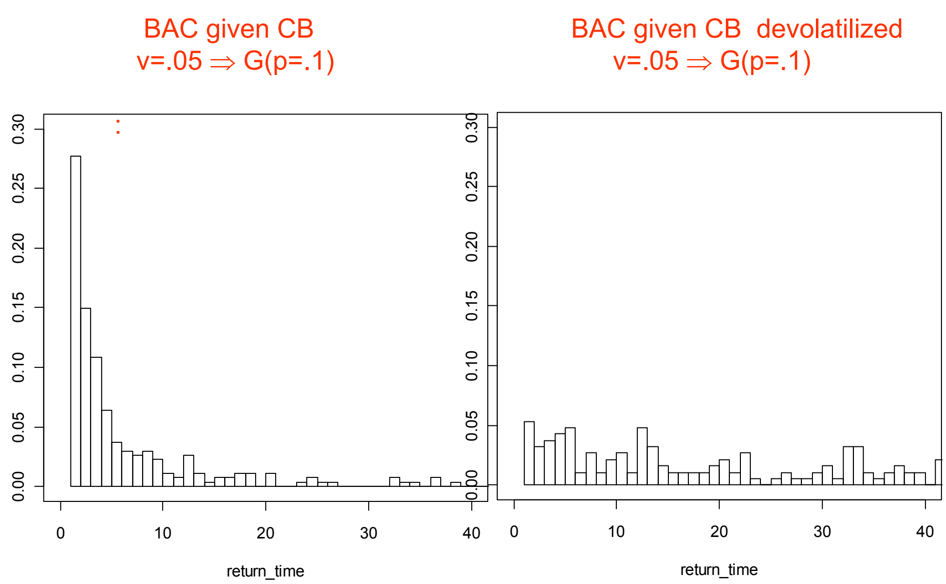
Answer: The estimated distribution for the return times is exactly the extremogram for specially chosen sets A & B. For example, in the upper tail case, $P(T_1 = 1)$ is estimated by

$$\hat{P}(T=1) = \frac{\sum_{t=1}^{n-1} I_{\{X_t \ge a_m, X_{t+1} \ge a_m\}}}{\sum_{t=1}^{n} I_{\{X_t \ge a_m\}}} = \frac{\#\text{consecutive pairs} > a_m}{\#\text{observations} > a_m}$$

$$\hat{\rho}_{A,B}(1) = \frac{\frac{m}{n}\sum_{t=1}^{n-1} I_{\{X_t \ge a_m, X_{t+1} \ge a_m\}}}{\frac{m}{n}\sum_{t=1}^{n} I_{\{X_t \ge a_m\}}}$$

Remark: So theory and methodology (permutation/bootstrapping) developed for the extremogram applies to the histogram

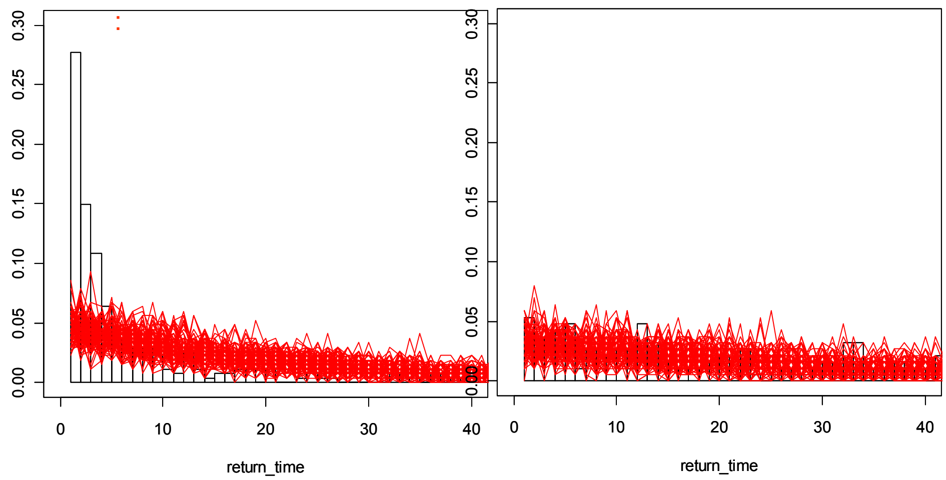
Bivariate Return Times (Citibank and Bank of America)



Bivariate Return Times (Citibank and Bank of America)

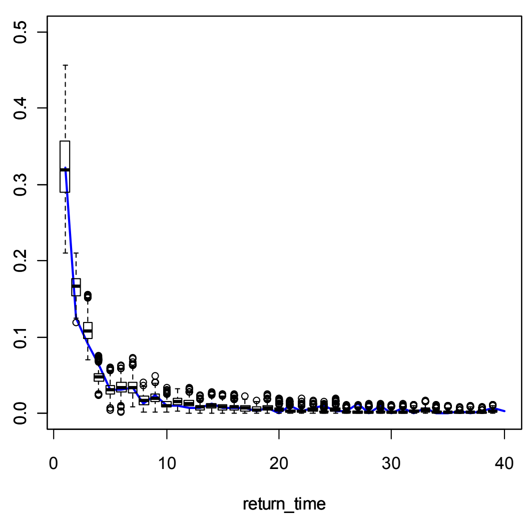
BAC given CB
v=.05 ⇒ G(p=.1)

BAC given CB devolatilized
v=.05 ⇒ G(p=.1)



Bootstrapping Return Times (BAC log-returns)

BAC, v=.05 ⇒ G(p=.1)



Wrap-up

- *Extremogram* is another potential tool for estimating extremal dependence that may be helpful for discriminating between models on the basis of extreme value behavior.
- *Regular variation* is a flexible tool for modeling both *dependence* and *tail heavyness.*
- Permutation procedures are a *quick* and *clean* way to test for significant values in the extremogram and other statistics.
- *Bootstrapping* may prove useful for constructing CI's for the extremogram and also for assessing extremal dependence.
- The *Extremogram* can provide insight on extremal dependence between components in a multivariate time series.
- Interesting connection between *return times* and the *extremogram.*
- *Extremogram* is a cool name!