In the name of God

On Gorenstein homological dimension of Groups

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In this talk, Γ is a group and $\mathbb{Z}\Gamma$ is its associated (integral) group ring. All considered modules - if not specified otherwise - are left $\mathbb{Z}\Gamma$-modules that, for simplicity, will be called (left) Γ-modules. This talk is based on a joint work with J. Asadollahi, A. Hajizamani and Sh. Salarian.

Definition . *A complete flat resolution is an exact sequence of flat* Γ*-modules*

$$\mathbf{F}_\bullet : \cdots \longrightarrow F_{i+1} \longrightarrow F_i \longrightarrow F_{i-1} \longrightarrow \cdots$$

such that $I \otimes_\Gamma \mathbf{F}_\bullet$ *is exact for any injective* Γ*-module* I*. A* Γ*-module* M *is called Gorenstein flat if it is a syzygy of a complete flat resolution, i.e., it is of the form* $M = \mathrm{Ker}(F_i \to F_{i-1})$*, for some integer* i*.*

It follows from the definition that $\mathrm{Tor}_i^\Gamma(I, M) = 0$ for all $i \geq 1$ and any injective Γ-module I.

Example . *1. Every flat module is Gorenstein flat.*

2. For any finite group Γ, $\mathbb{Z}$ as a Γ-module, with trivial action, is a Gorenstein flat Γ-module.

Definition . *Let M be a non-zero Γ-module. We say that Gorenstein flat dimension of M is $n \geq 0$, denoted $\mathrm{Gfd}_{\Gamma} M = n$, if n is the least integer for which there exists a proper flat resolution of M such that its nth syzygy is Gorenstein flat. If no such n exists, then we shall write $\mathrm{Gfd}_{\Gamma} M = \infty$. By convention, $\mathrm{Gfd}_{\Gamma} 0 = -\infty$.*

Definition . *For any group* Γ*, the Gorenstein homological dimension of* Γ*, denoted* $\operatorname{Ghd}\Gamma$*, is defined to be the Gorenstein flat dimension of the trivial* Γ*-module* $\mathbb{Z}$*; that is* $\operatorname{Ghd}\Gamma = \operatorname{Gfd}_\Gamma \mathbb{Z}$*. It is known that* $\operatorname{Ghd}\Gamma \leq \operatorname{hd}\Gamma$*, in addition equality holds provided that* $\operatorname{hd}\Gamma$ *is finite, where* $\operatorname{hd}\Gamma = \operatorname{fd}_\Gamma \mathbb{Z}$*.*

Example . *Let* Γ *be a locally free group. Then we have the following.*

(1) *Since* $\operatorname{hd}\Gamma = 1$, $\operatorname{Ghd}\Gamma = \operatorname{hd}\Gamma = 1$.

(2) *Set* $\Gamma'' = \Gamma \oplus \Gamma'$, *where* Γ' *is a finite group. Then* $\operatorname{hd}\Gamma'' = \infty$ *while* $\operatorname{Ghd}\Gamma'' = 1$.

Proposition . *Let* Γ *be a group. Then* $\operatorname{Ghd}\Gamma = 0$ *if and only if* Γ *is a finite group.*

Definition . *Let* Γ *be a group. We recall that* $\operatorname{sfli}\Gamma$ *is the supremum of the flat lengths of injective* Γ*-modules and* $\operatorname{silf}\Gamma$ *is the supremum of the injective lengths of flat* Γ*-modules.*

Moreover, it is straight forward to see that, for any group Γ,

$$\operatorname{sfli}\Gamma = \sup\{i : \operatorname{Tor}_i^{\Gamma}(-, I) \neq 0, \text{ for some injective } \Gamma-\text{module } I\}$$

and

$$\operatorname{silf}\Gamma = \sup\{i : \operatorname{Ext}_{\Gamma}^{i}(-, F) \neq 0, \text{ for some flat } \Gamma-\text{module } F\}.$$

Proposition . *Let* Γ *be any group. The following conditions are equivalent.*

(i) $\operatorname{sfli}\Gamma < \infty$.

(ii) $\operatorname{Gfd}_{\Gamma} M < \infty$, *for any* Γ*-module* M.

In particular, if $\operatorname{sfli}\Gamma$ *is finite, then for any* Γ*-module* M, $\operatorname{Gfd}_{\Gamma} M \leq \operatorname{sfli}\Gamma + 1$.

Lemma . *Let* Γ' *be a subgroup of* Γ *and* $F \to M$ *be a flat precover of the* Γ*-module* M*. Then* $F \to M$ *is also a flat precover of* M *as* Γ'*-module.*

Lemma . *Let* Γ' *be a subgroup of* Γ *of finite index and* M *be a Gorenstein flat* Γ*-module. Then* M *is Gorenstein flat as* Γ'*-module.*

Corollary . *Let* Γ' *be a subgroup of* Γ *of finite index. Then* $\operatorname{Ghd}\Gamma' \leq \operatorname{Ghd}\Gamma$.

Proposition . *Let* Γ *be a group with* $\mathrm{silf}\,\Gamma < \infty$. *If* $M \oplus F$ *is a Gorenstein flat* Γ*-module and* F *is a flat* Γ*-module, then* M *is a Gorenstein flat* Γ*-module.*

Proposition . *Let* Γ *be a group with* $\mathrm{silf}\,\Gamma$ *is finite and let* M *be a* Γ*-module such that* $\mathrm{Gfd}_{\Gamma} M \leq n$. *Then the* n*th syzygy of every proper flat resolution of* M *is a Gorenstein flat* Γ*-module.*

Theorem . *Let* Γ *be a group and* Γ' *be its subgroup of finite index. Then* $\operatorname{Ghd}\Gamma = \operatorname{Ghd}\Gamma'$ *provided that* $\operatorname{silf}\Gamma$ *is finite.*

Serre's Theorem. Let Γ be a torsion-free group. If Γ' is a subgroup of Γ of finite index, then $\operatorname{hd}\Gamma' = \operatorname{hd}\Gamma$.

Definition . *We recall the Ikenaga's generalized homological dimension of groups,* $\underline{hd}\,\Gamma$*, as follows:*

$\underline{hd}\,\Gamma = \sup\{i : \mathrm{Tor}_i^{\Gamma}(M, I) \neq 0,\ \ M \text{ is } \mathbb{Z}\text{−torsion free and } I \text{ is } \Gamma\text{−injective}\}$

Theorem . *(Emmanouil). For any group* Γ*,* $\underline{hd}\,\Gamma = 0$ *if and only if* Γ *is locally finite.*

Corollary . *For any group* Γ*,* $\mathrm{sfli}\,\Gamma = 1$ *if and only if* Γ *is locally finite.*

It is known that for any group Γ, $\mathrm{silf}\,\Gamma = \mathrm{silp}\,\Gamma = \mathrm{spli}\,\Gamma = 1$ if and only if Γ is a finite group.

Theorem . *(Bahlekeh, Dembegioti and Talelli). For any group Γ, $\mathrm{Gcd}\,\Gamma = \underline{cd}\,\Gamma$.*

We recall that $\underline{cd}\,\Gamma$ is the generalized cohomological dimension of Γ, defined by Ikenaga , as follows:

$\underline{cd}\,\Gamma = \sup\{i : \mathrm{Ext}^i_\Gamma(M, P) \neq 0$, where M is $\mathbb{Z}-$free and P is $\Gamma-$projective$\}$

Moreover, $\mathrm{Gcd}\,\Gamma := \mathrm{Gpd}_\Gamma\,\mathbb{Z}$ whereas $\mathbb{Z}$ is a trivial Γ-module.

Proposition . *Let* sfli Γ *is finite. Then every Gorenstein projective* Γ*-module is Gorenstein flat.*

Thank you all