# Linkage of Finite Gorenstein Dimension Modules

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# Introduction

The theory of linkage of algebraic varieties introduced by Peskine and Szpiro (1974).

Martsinkovsky and Strooker (2004) give its analogous definition for modules over non–commutative semiperfect Noetherian rings by using the composition of the two functors:

transpose and syzygy.

# Introduction

These functors and their compositions were studied by Auslander and Bridger in *"Stable module theory"* (1969).
The Gorenstein (or $G$-) dimension was introduced by Auslander (1966–7) and studied by Auslander and Bridger (1969).

In this work, we study the theory of linkage for class of modules which have finite Gorenstein dimensions. In particular, for a horizontally linked module $M$ of finite and positive $G$-dimension, we study the role of its reduced grade, r.grade $(M)$, on the depth of its linked module $\lambda M$.

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# Notations and Definitions (semiperfect)

Let $R$ be a ring. Consider an R-modules M with a submodule N. The module M is said to be a *superfluous extension* of $N$ if for every submodule H of M, $H + N = M \Longrightarrow H = M$.

# Notations and Definitions (semiperfect)

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Let $X$ be an $R$–module. A projective cover of X is a pair $(P, f)$ such that $P$ is a projective $R$–module and $f : P \longrightarrow X$ is an epimorphism with $P$ is a superfluous extension of $\ker f$.

# Notations and Definitions (semiperfect)

Projective covers and their superfluous epimorphisms, when they exist, are unique up to isomorphism. The main effect of $f$ having a superfluous kernel is the following: if $K$ is any proper submodule of $P$, then $f(K) \neq X$. If $(P, f)$ is a projective cover of $M$, and $P'$ is another projective module with an epimorphism $f' : P' \longrightarrow X$, then there is an epimorphism $\alpha$ from $P'$ to $P$ such that $f\alpha = f'$.

Unlike injective envelopes, which exist for every left (right) R-module regardless of the ring R, left (right) R-modules do not in general have projective covers.

A ring R is called left (right) perfect if every left (right) R-module has a projective cover.

# Notations and Definitions

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## Commutative semiperfect noetherian rings

Throughout, $R$ is a commutative semiperfect noetherian ring and all modules are finite (i.e. finitely generated) $R$–modules so that any such module has a projective cover.

# Notations and Definitions

## Transpose

Let

$$P_1 \xrightarrow{f} P_0 \to M \to 0$$

be a finite projective presentation of $M$. The *transpose* of $M$, $\operatorname{Tr} M$, is Coker $f^*$, where $(-)^* := \operatorname{Hom}_R(-, R)$, which satisfies in the exact sequence

$$0 \to M^* \to P_0^* \xrightarrow{f^*} P_1^* \to \operatorname{Tr} M \to 0.$$

and is unique up to projective equivalence; that is if $P \oplus M = Q \oplus M'$ (denoted by $\underline{M} = \underline{M'}$) with $P$ and $Q$ are projective then $\operatorname{Tr} M \cong \operatorname{Tr} M'$. A *stable* module $M$ is a module which has no non–trivial projective summands.

# Notations and Definitions

This is a result from Auslander (1965) that there is an exact sequence

$$0 \longrightarrow \operatorname{Ext}^1_R(\operatorname{Tr} M, R) \longrightarrow M \xrightarrow{e_M} M^{**} \longrightarrow \operatorname{Ext}^2_R(\operatorname{Tr} M, R) \longrightarrow 0.$$

where $e_M : M \to M^{**}$ is the natural map.

## syzygy

Let $P \xrightarrow{\alpha} M$ be an epimorphism such that $P$ is a projective. The syzygy module of $M$, denoted by $\Omega M$, is the kernel of $\alpha$ which is unique up to projective equivalence. Thus $\Omega M$ is uniquely determined, up to isomorphism, by a projective cover of $M$.

# Notations and Definitions

**Linkage of ideals**

Let $R$ be Gorenstein local and let $\mathfrak{c}$ be an ideal of $R$ such that $R/\mathfrak{c}$ is Gorenstein. Two ideals $\mathfrak{a}$ and $\mathfrak{b}$ of $R$ are said to be linked by $\mathfrak{c}$ if $\mathfrak{c} \subseteq \mathfrak{a} \cap \mathfrak{b}$, $\mathfrak{a} = \mathfrak{c} : \mathfrak{b}$ and $\mathfrak{b} = \mathfrak{c} : \mathfrak{a}$.

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Martsinkovsky and Strooker (MS) have introduced the operator $\lambda := \Omega \mathrm{Tr}$.

They showed that over a ring $R$, the ideals $\mathfrak{a}$ and $\mathfrak{b}$ are linked by zero ideal if and only if $R/\mathfrak{a} \cong \lambda(R/\mathfrak{b})$ and $R/\mathfrak{b} \cong \lambda(R/\mathfrak{a})$.

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(MS): Two $R$–modules $M$ and $N$ are said to be *horizontally linked* if $M \cong \lambda N$ and $N \cong \lambda M$.

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In this situation the module $M$ is called a *horizontally linked module* and one has $M \cong \lambda^2 M$.

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## Definition

*An $R$–module $M$ is said to belong to the G-class, $G(R)$, whenever*

(i) *the biduality map $e_M : M \to M^{**}$ is an isomorphism;*

(ii) *$Ext^i_R(M, R) = 0$ for all $i > 0$;*

(iii) *$Ext^i_R(M^*, R) = 0$ for all $i > 0$.*

# Notations and Definitions

## $G$–dimension

Any projective module is in $G$-class. Trivially any $R$–module $M$ has a G-resolution which is a right acyclic complex of modules in $G(R)$ whose 0th homology module is $M$. The module $M$ is said to have finite $G$-dimension, denoted by $\operatorname{Gdim}_R(M)$, if it has a $G$-resolution of finite length. Note that $\operatorname{Gdim}_R(M) \leq \operatorname{pd}_R(M)$.

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## Theorem

*(Masiek) If Gdim* $_R(M) < \infty$*, then*

*(i)* $\operatorname{Gdim}_R(M) = \sup\{i \geq 0 \mid \operatorname{Ext}^i_R(M, R) \neq 0\}$*; and (ii) if R is local, then* $\operatorname{Gdim}_R(M) = \operatorname{depth} R - \operatorname{depth}_R(M)$ *(Auslander-Bridger).*

# Linkage and Reduced Grade

**Definition**

*The reduced grade of an R–module M is defined to be*

$$r.grade\,(M) = \inf\{i > 0 \mid Ext^i_R(M,R) \neq 0\},$$

*introduced by Hoshino (1990).*

Note that $\text{grade}_R(M) = \text{r.grade}_R(M)$ if $\text{grade}_R(M) > 0$. Moreover, if $\text{Gdim}_R(M) = 0$ then $\text{r.grade}\,(M) = \infty$. For modules of finite and positive G-dimension, one has $\text{r.grade}\,(M) \leq \text{Gdim}_R(M)$ and so it is finite.

# Linkage and Reduced Grade

## Lemma

*Let $M$ be a horizontally linked $R$–module of finite and positive G-dimension. Set $n = r.grade(M)$. Then*

$$Ass_R(Ext^n_R(M,R)) = \{\mathfrak{p} \in Spec\, R \mid Gdim_{R_\mathfrak{p}}(M_\mathfrak{p}) \neq 0, depth_{R_\mathfrak{p}}((\lambda M)_\mathfrak{p}) = n = r.grade_{R_\mathfrak{p}}(M_\mathfrak{p})\}.$$

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## Proposition

*Let $M$ be a horizontally linked $R$–module of finite G-dimension. Then $Gdim_R(M) = 0$ if and only if $depth_{R_\mathfrak{p}}(M_\mathfrak{p}) + depth_{R_\mathfrak{p}}((\lambda M)_\mathfrak{p}) > depth\,R_\mathfrak{p}$ for all $\mathfrak{p} \in Spec\,R \setminus X^0(R)$.*

Here $X^i(R) = \{\mathfrak{p} \in \mathrm{Spec}\,R \mid \mathrm{depth}\,R_\mathfrak{p} \leq i\}$.

# Linkage and Reduced Grade

## Corollary

*Let M be a horizontally self-linked R–module of finite G-dimension. Then* $\mathrm{Gdim}_R(M) = 0$ *if and only if* $\mathrm{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) > \frac{1}{2}(\mathrm{depth}\, R_{\mathfrak{p}})$ *for all* $\mathfrak{p} \in \mathrm{Spec}\, R \setminus X^0(R)$.

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## Corollary

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## Proposition

*Let $M$ be a horizontally linked $R$–module of finite and positive G-dimension. Set $t_M = \operatorname{r.grade}(M) + \operatorname{r.grade}(\lambda M)$, then $M$ is of G-dimension zero on $X^{t_M - 1}(R)$.*

## Linkage and Reduced Grade

Auslander shows that, for a positive integer $k$, the following statements are equivalent.

(i) $r.grade\,(Tr\,M) > k$.

(ii) $M$ is a $k$th syzygy.

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**Proposition**

*Let $M$ be a horizontally linked $R$–module of finite G-dimension. Let $k$ be a positive integer. Then the following statements are equivalent.*

(i) $r.grade\,(M) \geq k$.

(ii) $\lambda M$ *is a* $k$*th syzygy.*

# Linkage and Reduced Grade

## Proposition

*Let M be an R–modules. If M is horizontally linked module, then*

$$Ext^i_R(M, M) \cong Ext^i_R(\lambda M, \lambda M)$$

*for all i,* $1 \leq i < \inf\{r.grade\,(M), r.grade\,(\lambda M)\}$.

*In particular, if* $Gdim_R(M) = 0$ *then* $Ext^i_R(M, M) \cong Ext^i_R(\lambda M, \lambda M)$ *for all* $i > 0$.

# Reduced $G$–Perfect Modules

Let $M$ be an $R$–module of finite positive $G$–dimention. The following inequalities are just mentioned:

$$\operatorname{grade}_R(M) \leq \operatorname{r.grade}(M) \leq \operatorname{Gdim}_R(M) \leq \operatorname{pd}_R(M).$$

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# Reduced $G$–Perfect Modules

**Theorem**

*Let $(R, \mathfrak{m})$ be a local Cohen-Macaulay ring of dimension d. If M is reduced G-perfect of G-dimension n, then*
$\mathrm{depth}_R(M) + \mathrm{depth}_R(\lambda M) = d + \mathrm{depth}_R(\mathrm{Ext}^n_R(M, R))$.

**Proposition**

*Let M be a reduced G-perfect R–module of G-dimension n, then the following statements hold true.*

(i) $\mathrm{Ext}^i_R(\lambda M, R) \cong \mathrm{Ext}^{n+i}_R(\mathrm{Ext}^n_R(M, R), R)$ *for all* $i > 0$.

(ii) *Assume that M is stable R–module. Then M is horizontally linked if and only if* $\mathrm{r.grade}(M) + \mathrm{r.grade}(\lambda M) = \mathrm{grade}_R(\mathrm{Ext}^n_R(M, R))$.

# Reduced $G$–Perfect Modules

Let $R$ be local. Recall from Evan-Griffith "*Syzygies*" (LMS Lecture Notes 1985) that

$$\operatorname{syz}(M) =$$
$$\operatorname{Sup}\{n \mid M \text{ is } n\text{th syzygy in a minimal free resolution of an } R\text{–module } N\}.$$

Note that $\operatorname{syz}(M) = \infty$, whenever $\operatorname{Gdim}_R(M) = 0$. If $M$ is a horizontally linked of finite and positive $G$-dimension then we have $\operatorname{syz}(M) = \operatorname{r.grade}(\lambda M)$.

# Reduced $G$–Perfect Modules

**Theorem**

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- $\operatorname{Ext}_R^{\operatorname{r.grade}(\lambda M)}(\lambda M, R) \cong \operatorname{Ext}_R^d(\operatorname{Ext}_R^{\operatorname{Gdim}_R(M)}(M, R), R)$.

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# Linkage and Local Cohomology

Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals in a Gorenstein local ring $R$ which are linked by a Gorenstein ideal $\mathfrak{c}$. Schenzel (1983) proved that the Serre condition $(S_r)$ for $R/\mathfrak{a}$ is equivalent to the vanishing of the local cohomology groups $\mathrm{H}^i_{\mathfrak{m}}(R/\mathfrak{b}) = 0$ for all $i$, $\dim(R/\mathfrak{b}) - r < i < \dim(R/\mathfrak{b})$. Here we extend this result for any horizontally linked module of finite $G$-dimension over a Cohen-Macaulay local ring.

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First we bring the following lemma which is clear if the ground ring is Gorenstein by using the Local Duality Theorem.

# Linkage and Local Cohomology

# Linkage and Local Cohomology

## Lemma

*Let $R$ be a Cohen-Macaulay local ring of dimension $d$ and let $M$ be an $R$–module of dimension $d$ which is not maximal Cohen-Macaulay. If $\mathrm{Gdim}_R(\lambda M) < \infty$ then $\sup\{i \mid H^i_{\mathfrak{m}}(M) \neq 0, i \neq d\} = d - \mathrm{r.grade}(M)$.*

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Now we are able to generalize a result of Schenzel (1983) for modules of finite Gorenstein dimension.

## Theorem

*Let R be a Cohen-Macaulay local ring of dimension $d$, and let $M$ be a horizontally linked $R$–module of finite G-dimension. Let $k$ be a non-negative integer. Then $M$ satisfies the Serre condition $(S_k)$ if and only if $H^i_{\mathfrak{m}}(\lambda M) = 0$ for all $i$, $d - k + 1 \leq i \leq d - 1$.*

# Linkage and Local Cohomology

D-Gheibi-Hassanzadeh-Sadeghi (2011) have shown that $\operatorname{Ext}^i_R(M, R) \cong \mathrm{H}^i_{\mathfrak{m}}(\lambda M)$ for all $i$, $1 \leq i < \dim R$ whenever $R$ is Cohen-Macaulay with canonical module $\omega_R$, $\operatorname{Tor}^R_i(M, \omega_R) = 0$ for all $i > 0$ and $\operatorname{Ext}^i_R(M, R)$ is of finite length for all $i$, $1 \leq i < \dim R$.

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**Theorem**

*Let $R$ be a local ring with $\operatorname{depth} R \geq 2$ and let $M$ be an $R$–module. Assume that $n$ is an integer such that $1 < n \leq \operatorname{depth} R$ and that $\operatorname{Ext}^i_R(M,R)$ is of finite length for all $i$, $1 \leq i < n$. Then $\operatorname{Ext}^i_R(M,R) \cong H^i_{\mathfrak{m}}(\lambda M)$ for all $i$, $1 \leq i < n$.*

# Semidualizing Modules and Evenly Linked Modules

## Definition

*An $R$–module $M$ is said to be linked to an $R$–module $N$, by an ideal $\mathfrak{c}$ of $R$, if $\mathfrak{c} \subseteq \mathrm{Ann}_R(M) \cap \mathrm{Ann}_R(N)$ and $M$ and $N$ are horizontally linked as $R/\mathfrak{c}$–modules. In this situation we denote $M \underset{\mathfrak{c}}{\sim} N$.*

Let $(R, \mathfrak{m})$ be a Gorenstein local ring, $\mathfrak{c}_1$ and $\mathfrak{c}_2$ Gorenstein ideals. Let $M_1, M$ and $M_2$ be $R$–modules such that $M_1$ is linked to $M$ by $\mathfrak{c}_1$ and $M$ is linked to $M_2$ by $\mathfrak{c}_2$. Martsinkovsky and Strooker prove that $\mathrm{Gdim}_R(M_1) = \mathrm{Gdim}_R(M_2)$ and also

$$\mathrm{Ext}^i_{R/\mathfrak{c}_1}(M_1, R/\mathfrak{c}_1) \cong \mathrm{Ext}^i_{R/\mathfrak{c}_2}(M_2, R/\mathfrak{c}_2) \text{ for all } i > 0.$$

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Let $(R, \mathfrak{m})$ be a Gorenstein local ring, $\mathfrak{c}_1$ and $\mathfrak{c}_2$ Gorenstein ideals. Let $M_1, M$ and $M_2$ be $R$–modules such that $M_1$ is linked to $M$ by $\mathfrak{c}_1$ and $M$ is linked to $M_2$ by $\mathfrak{c}_2$. Martsinkovsky and Strooker prove that $\operatorname{Gdim}_R(M_1) = \operatorname{Gdim}_R(M_2)$ and also

$$\operatorname{Ext}^i_{R/\mathfrak{c}_1}(M_1, R/\mathfrak{c}_1) \cong \operatorname{Ext}^i_{R/\mathfrak{c}_2}(M_2, R/\mathfrak{c}_2) \text{ for all } i > 0.$$

# Semidualizing Modules and Evenly Linked Modules

In this part we establish this isomorphism, without assuming $R$ is Gorenstein, but we assume some conditions on the modules $M_1$, $M$, $M_2$ and on ideals $\mathfrak{c}_1$, $\mathfrak{c}_2$.

Throughout this section $R$ is a local ring, $K$ and $M$ are $R$–modules. Denote $M^{\dagger} = \operatorname{Hom}_R(M, K)$. The module $M$ is called $K$*-reflexive* if the canonical map $M \to M^{\dagger\dagger}$ is bijective.

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A $G_K$- resolution of a finite $R$–module $M$ is a right acyclic complex of modules of $G_K$- dimensions zero whose 0th homology module is $M$. The module $M$ is said to have finite $G_K$- dimension, denoted by $G_K\text{-dim}_R(M)$, if it has a $G_K$- resolution of finite length.

# Semidualizing Modules and Evenly Linked Modules

**Definition**

*An $R$–module $K$ is called a semidualizing module (suitable), if*

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Semidualizing modules are studied by Foxby, Golod, and many others. It is obvious that $R$ itself is a semidualizing $R$–module. Also it is well known that if $R$ is Cohen-Macaulay then its canonical module (if exists) is a semidualizing module.

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Let $K$ be a semidualizing $R$–module. For an $R$–module $M$ of finite $G_K$- dimension the following statements hold true.

(i) $G_K\text{-}dim_R(M) = \sup\{i \mid Ext^i_R(M, K) \neq 0, i \geq 0\}$.

(ii) If $G_K$-dim $_R(M) < \infty$ then $G_K\text{-}dim_R(M) = depth\, R - depth_R(M)$.

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# Semidualizing Modules and Evenly Linked Modules

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*An $R$–module $M$ is called $G_K$-perfect if $\text{grade}_R(M) = G_K\text{-dim}_R(M)$. An ideal $I$ is called $G_K$-perfect if $R/I$ is $G_K$-perfect as $R$–module. An $R$–module $M$ is called $G_K$-Gorenstein if $M$ is $G_K$-perfect and $\text{Ext}^n_R(M, K)$ is cyclic, where $n = G_K\text{-dim}_R(M)$. An ideal $I$ is called $G_K$-Gorenstein if $R/I$ is $G_K$-Gorenstein as $R$–module.*

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Note that if $K$ is a semidualizing $R$–module and $I$ is a $G_K$-Gorenstein ideal of $G_K$-dimension $n$, then $\text{Ext}^n_R(R/I, K) \cong R/I$. (Golod)

# Semidualizing Modules and Evenly Linked Modules

**Proposition**

*Let $K$ be a semidualizing $R$–module, $\mathfrak{c}_1$ and $\mathfrak{c}_2$ two $G_K$-Gorenstein ideals. Assume that $M_1$, $M$, and $M_2$ are $R$–modules such that $M_1 \underset{\mathfrak{c}_1}{\sim} M$ and $M \underset{\mathfrak{c}_2}{\sim} M_2$. Denote the common value of grade$(\mathfrak{c}_1)$ and grade$(\mathfrak{c}_2)$ by $n$. Then $\operatorname{Ext}^i_R(M_1, K) \cong \operatorname{Ext}^i_R(M_2, K)$ for all $i$, $i > n$.*

# Semidualizing Modules and Evenly Linked Modules

## Corollary

*Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with canonical module $\omega_R$. Assume that $\mathfrak{c}_1$ and $\mathfrak{c}_2$ are Gorenstein ideals and that $M_1$,$M$, and $M_2$ are R–modules such that $M_1 \underset{\mathfrak{c}_1}{\sim} M$ and $M \underset{\mathfrak{c}_2}{\sim} M_2$. Set $n = \dim_R(M_1) = \dim_R(M_2)$. Then $H^i_{\mathfrak{m}}(M_1) \cong H^i_{\mathfrak{m}}(M_2)$, for all i, $i < n$.*

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# MOTASHAKERAM