On an endomorphism ring of local cohomology

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It is based on a joint work with Peter Schenzel
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Why local cohomolgy

Local cohomology was invented by Grothendieck in 1960s to prove some theorems in algebraic geometry. It has many applications in topology, geometry, combinatorics, and computational subjects.

Definition

Let M be an R-module and $\mathfrak{a}$ an ideal of R, then we define i-th local cohomology module as $H^i_{\mathfrak{a}}(M) = \varinjlim \operatorname{Ext}^i_R(R/\mathfrak{a}^n, M)$.

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Hochster-Huneke (1994)

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Hellus-Stückrad (2007), Hellus-Schenzel(2008), Schenzel(2009)

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Some questions in local cohomology

Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and $E_R(R/\mathfrak{m})$ be the injective hull of the residue field $R/\mathfrak{m}$ and M a finitely generated R-module of dimension d:

1. How one can express $H^d_{\mathfrak{a}}(M)$ via $H^d_{\mathfrak{m}}(M)$.
2. What are the properties of $\operatorname{Hom}_R(H^{\dim R}_{\mathfrak{a}}(R), E_R(R/\mathfrak{m}))$.
3. What are the properties of $\operatorname{Hom}_{\widehat{R}}(H^{\dim R}_{\mathfrak{a}}(R), H^{\dim R}_{\mathfrak{a}}(R))$.
4. What are some applications of the above questions?

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To Control top local cohomology

Express $H^d_{\mathfrak{a}}(M)$ via $H^d_{\mathfrak{m}}(M)$.

Explanation

Put $d := \dim M$. When $H^d_{\mathfrak{a}}(M) \neq 0$ one of the most important views concerning this is to express $H^d_{\mathfrak{a}}(M)$ via $H^d_{\mathfrak{m}}(M)$. More precisely the kernel of the natural epimorphism $H^{\dim M}_{\mathfrak{m}}(M) \to H^{\dim M}_{\mathfrak{a}}(M)$ has been calculated explicitly.

Note

For an R-module M let $0 = \cap_{i=1}^{n} Q_i(M)$ denote a minimal primary decomposition of the zero submodule of M. That is $M/Q_i(M)$, $i = 1, \dots, n$, is a $\mathfrak{p}_i$-primary R-module. Clearly $\operatorname{Ass}_R M = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$.

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Some Definitions

Definition

Let $\mathfrak{a} \subset R$ denote an ideal of R. We define two disjoint subsets U, V of $\operatorname{Ass}_R M$ related to $\mathfrak{a}$

(a) $U = \{\mathfrak{p} \in \operatorname{Ass}_R M | \dim R/\mathfrak{p} = d \text{ and } \dim R/\mathfrak{a} + \mathfrak{p} = 0\}$.

(b) $V = \{\mathfrak{p} \in \operatorname{Ass}_R M | \dim R/\mathfrak{p} < d \text{ or } \dim R/\mathfrak{p} = d \text{ and } \dim R/\mathfrak{a} + \mathfrak{p} > 0\}$.

Finally we define $Q_{\mathfrak{a}}(M) = \cap_{\mathfrak{p}_i \in U} Q_i(M)$. In the case $U = \emptyset$, put $Q_{\mathfrak{a}}(M) = M$.

Definition

Let M denote a finitely generated module over the local ring $(R, \mathfrak{m})$. Let $\mathfrak{a} \subset R$ denote an ideal. Then define $P_{\mathfrak{a}}(M)$ as the intersection of all the primary components of $\operatorname{Ann}_R M$ such that $\dim R/\mathfrak{p} = \dim M$ and $\dim R/\mathfrak{a} + \mathfrak{p} = 0$. Clearly $P_{\mathfrak{a}}(M)$ is the pre-image of $Q_{\mathfrak{a}R/\operatorname{Ann}_R M}(R/\operatorname{Ann}_R M)$ in R.

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Main Theorem

Theorem

Let $\mathfrak{a}$ denote an ideal of a local ring $(R, \mathfrak{m})$. Let M be a finitely generated R-module and $d = \dim M$. Then there is a natural isomorphism

$$H^d_{\mathfrak{a}}(M) \cong H^d_{\mathfrak{m}\widehat{R}}(\widehat{M}/Q_{\mathfrak{a}\widehat{R}}(\widehat{M})) \cong H^d_{\mathfrak{m}\widehat{R}}(\widehat{M}/P_{\mathfrak{a}}(\widehat{M})\widehat{M}).$$

Proof

Using the short exact sequence

$$0 \to Q_{\mathfrak{a}}(M) \to M \to M/Q_{\mathfrak{a}}(M) \to 0$$

and applying local cohomology module to it we prove the claim. to this end note that $\operatorname{Ass}_R Q_{\mathfrak{a}}(M) = V$, $\operatorname{Ass}_R M/Q_{\mathfrak{a}}(M) = U$ and $U \cup V = \operatorname{Ass}_R M$.

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Notation

For an ideal $\mathfrak{a} \subset R$ with $\dim R/\mathfrak{a} = d$ we will denote by $\mathfrak{a}_d$ the intersection of those primary components in a minimal reduced primary decomposition of $\mathfrak{a}$ which are of dimension d.

Notation and Definition

For a local ring $(R, \mathfrak{m})$ which is a factor ring of a Gorenstein ring $(S, \mathfrak{n})$ with $r = \dim S$. Then there are functorial isomorphisms

$$H^d_{\mathfrak{m}}(M) \cong \mathrm{Hom}_R(\mathrm{Ext}^{r-d}_S(M, S), E(R/\mathfrak{m})), d := \dim M,$$

where M is a finitely generated R-module. The module $K_M = \mathrm{Ext}^{r-d}_S(M, S)$ is called the canonical module of M.

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Lemma

Let $\mathfrak{a}$ denote an ideal in a d-dimensional local ring $(R, \mathfrak{m})$. Then

(1) $T_{\mathfrak{a}}(R) = \operatorname{Hom}_R(H^d_{\mathfrak{a}}(R), E_R(R/\mathfrak{m}))$ is a finitely generated $\widehat{R}$-module.

(2) $\operatorname{Ass}_{\widehat{R}} T_{\mathfrak{a}}(R) = \{\mathfrak{p} \in \operatorname{Ass} \widehat{R} | \dim \widehat{R}/\mathfrak{p} = \dim R \text{ and } \dim \widehat{R}/\mathfrak{a}\widehat{R} + \mathfrak{p} = 0\}$.

(3) $K_{\widehat{R}}(\widehat{R}/Q_{\mathfrak{a}}(\widehat{R})) \cong T_{\mathfrak{a}}(R)$. In particular, It satisfies the S_2 condition. Furthermore when $\widehat{R}/Q_{\mathfrak{a}}(\widehat{R})$ is Cohen-Macaulay then so is $T_{\mathfrak{a}}(R)$.

Applications

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Lemma

(4) $\mathrm{Ann}_{\widehat{R}}(H^d_{\mathfrak{a}}(R)) = Q_{\mathfrak{a}}(\widehat{R})$.

Theorem

Let $\mathfrak{a}$ denote an ideal in a local ring $(R, \mathfrak{m})$. Let

$$\Phi : \widehat{R} \to \mathrm{Hom}_{\widehat{R}}(H^d_{\mathfrak{a}}(R), H^d_{\mathfrak{a}}(R))$$

the natural homomorphism. Then

(1) $\ker \Phi = Q_{\mathfrak{a}\widehat{R}}(\widehat{R})$.

(2) Φ is surjective if and only if $\widehat{R}/Q_{\mathfrak{a}\widehat{R}}(\widehat{R})$ satisfies S_2.

(3) $\mathrm{Hom}_{\widehat{R}}(H^d_{\mathfrak{a}}(R), H^d_{\mathfrak{a}}(R))$ is a finitely generated $\widehat{R}$-module.

(4) $\mathrm{Hom}_{\widehat{R}}(H^d_{\mathfrak{a}}(R), H^d_{\mathfrak{a}}(R))$ is a commutative semi-local Noetherian ring.

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Theorem

Let $\mathfrak{a}$ be an ideal of a complete local ring $(R, \mathfrak{m})$. For an integer $r \geq 2$ we have the following statements:

(1) Suppose $R/Q_{\mathfrak{a}}(R)$ has S_2. Then $T_{\mathfrak{a}}(R)$ satisfies the condition S_r if and only if $H^i_{\mathfrak{m}}(R/Q_{\mathfrak{a}}(R)) = 0$ for $d - r + 2 \leq i < d$.

(2) $R/Q_{\mathfrak{a}}(R)$ satisfies the condition S_r if and only if $H^i_{\mathfrak{m}}(T_{\mathfrak{a}}(R)) = 0$ for $d - r + 2 \leq i < d$ and $R/Q_{\mathfrak{a}}(R) \cong \operatorname{Hom}_R(H^d_{\mathfrak{a}}(R), H^d_{\mathfrak{a}}(R))$.

In particular, if $R/Q_{\mathfrak{a}}(R)$ has S_2 it is a Cohen-Macaulay ring if and only if the module $T_{\mathfrak{a}}(R)$ is Cohen-Macaulay.

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Some connectedness results

Hartshorne

Let $(R, \mathfrak{m})$ denote a local ring such that depth $R \geq 2$. Then Spec $R \setminus \{\mathfrak{m}\}$ is connected in Zariski topology.

Example

Let $R := k[x, y, u, v]/((x, y) \cap (u, v))$. Then R is a two dimensional ring such that Spec $R \setminus \{\mathfrak{m}\}$ is disconnected. Then R can not be Cohen-Macaulay ring.

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Let $(R, \mathfrak{m})$ denote a local ring. We denote by $\mathbb{G}(R)$ the undirected graph whose vertices are primes $\mathfrak{p} \in \operatorname{Spec} R$ such that $\dim R = \dim R/\mathfrak{p}$, and two distinct vertices $\mathfrak{p}, \mathfrak{q}$ are joined by an edge if and only if $(\mathfrak{p}, \mathfrak{q})$ is an ideal of height one.

Proposition

Let $(R, \mathfrak{m})$ denote a local ring and $d = \dim R$. Then the following conditions are equivalent:

(1) The graph $\mathbb{G}(R)$ is connected.

(2) $\operatorname{Spec} R/0_d$ is connected in codimension one.

(3) For every ideal $JR/0_d$ of height at least two, $\operatorname{Spec}(R/0_d) \setminus V(JR/0_d)$ is connected.

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Hochster-Huneke

Let $(R, \mathfrak{m})$ be a complete local equidimensional ring and $d = \dim R$. Then the following conditions are equivalent:

(1) $H^d_{\mathfrak{m}}(R)$ is indecomposable.

(2) K_R, the canonical module of R is indecomposable.

(3) The ring $\mathrm{Hom}_R(K_R, K_R)$ is local.

(4) For every ideal J of height at least two, $\mathrm{Spec}(R) \setminus V(J)$ is connected.

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The extension of Hochster-Huneke Theorem

Let $(R, \mathfrak{m})$ denote a complete local ring and $d = \dim R$. For an ideal $\mathfrak{a} \subset R$ the following conditions are equivalent:

(1) $H^d_{\mathfrak{a}}(R)$ is indecomposable.

(2) $\mathrm{Hom}_R(H^d_{\mathfrak{a}}(R), E(R/\mathfrak{m}))$ is indecomposable.

(3) The endomorphism ring of $H^d_{\mathfrak{a}}(R)$ is a local ring.

(4) The graph $\mathbb{G}(R/Q_{\mathfrak{a}}(R))$ is connected,

The extension of Hochster-Huneke Theorem

Let $(R, \mathfrak{m})$ denote a complete local ring and $d = \dim R$. For an ideal $\mathfrak{a} \subset R$ the following conditions are equivalent:

(1) $H^d_{\mathfrak{a}}(R)$ is indecomposable.

(2) $\operatorname{Hom}_R(H^d_{\mathfrak{a}}(R), E(R/\mathfrak{m}))$ is indecomposable.

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Number of connected components

Notation

We describe t, the number of connected components of $\mathbb{G}(R/Q_{\mathfrak{a}}(R))$.

Definition

A connected component of an undirected graph is a subgraph in which any two vertices are connected to each other by paths, and which is connected to no additional vertices.

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Let $\mathfrak{a}$ be an ideal in a local ring $(R, \mathfrak{m})$. Suppose that $Q = Q_{\mathfrak{a}}(R)$ is a proper ideal. Let $\mathbb{G}_i, i = 1, \ldots, t$, denote the connected components of $\mathbb{G}(R/Q)$. Let $Q_i, i = 1, \ldots, t$, denote the intersection of all $\mathfrak{p}$-primary components of a reduced minimal primary decomposition of Q such that $\mathfrak{p} \in \mathbb{G}_i$. Then $Q = \cap_{i=1}^{t} Q_i$ and $\mathbb{G}(R/Q_i) = \mathbb{G}_i, i = 1, \ldots, t$, is connected. Moreover, let $\mathfrak{a}_i, i = 1, \ldots, t$, denote the image of the ideal $\mathfrak{a}$ in R/Q_i.

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Theorem

Let $\mathfrak{a}$ denote an ideal of a complete local ring $(R, \mathfrak{m})$ with $d = \dim R \geq 2$. Then

$$\operatorname{End} H^d_{\mathfrak{a}}(R) \simeq \operatorname{End} H^d_{\mathfrak{a}_1}(R/Q_1) \times \ldots \times \operatorname{End} H^d_{\mathfrak{a}_t}(R/Q_t)$$

is a semi-local ring, $\operatorname{End} H^d_{\mathfrak{a}_i}(R/Q_i), i = 1, \ldots, t$, is a local ring and therefore t is equal to the number of maximal ideals of $\operatorname{End} H^d_{\mathfrak{a}}(R)$.

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THANK YOU VERY MUCH