In the name of God

New homological invariants for modules over group rings

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Table of Contents

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Let Γ be a group and M a Γ-module. Let $\mathbb{Z}$ is regarded as an Γ-module with trivial action.

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Γ admits a *complete resolution of projectives* $(\mathbf{P}'_\bullet, \mathbf{P}_\bullet, n)$ if there is an acyclic complex $\mathbf{P}'_\bullet$ of projective modules and a projective resolution $\mathbf{P}_\bullet$ of $\mathbb{Z}$ such that $\mathbf{P}'_\bullet$ and $\mathbf{P}_\bullet$ coincide in sufficiently high dimensions as follows:

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Two special groups

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Let us list two classes of groups for them the invariants silf and sfli are both finite.

- It is shown by Cornick and Kropholler that if Γ is an $H\mathcal{F}$-group of type FP_∞, then silp Γ is finite. So for this class of groups, silf Γ and sfli Γ are both finite.

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