Local cohomology modules and derived functors

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Some part of this talk is a joint work with Dr. F.Khosh-Ahang.
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General notations and terminology

1. R: Commutative Noetherian ring with non-zero identity
2. $\mathfrak{a}$: An ideal of R
3. M: An R-module
4. $\mathbb{N}_0$ (resp. $\mathbb{N}$): The set of non-negative (resp.) positive integers.

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Recall that:

For each R-module M, set $\Gamma_{\mathfrak{a}}(M) := \bigcup_{n\in\mathbb{N}}(0 :_M \mathfrak{a}^n)$
Also for a homomorphism $f : M \longrightarrow N$ of R-modules, we set $\Gamma_{\mathfrak{a}}(f)$ is the restriction of f to $\Gamma_{\mathfrak{a}}(M)$. Note that $f(\Gamma_{\mathfrak{a}}(M)) \subseteq \Gamma_{\mathfrak{a}}(N)$. Thus $\Gamma_{\mathfrak{a}}(-)$ becomes a covariant, R-linear, left exact functor from the category of R-modules and R-homomorphisms to itself. We call $\Gamma_{\mathfrak{a}}(-)$ the $\mathfrak{a}$-torsion functor. For $i \geq 0$, the i-th right derived functor of $\Gamma_{\mathfrak{a}}(-)$ is denoted by $H^i_{\mathfrak{a}}(-)$ and will be referred to as the i-th local cohomology functor with respect to $\mathfrak{a}$.

Definition:

There is a canonical map

$$\mu_M : R \longrightarrow \mathrm{End}_R(M)$$

such that for $r \in R$, $\mu_M(r)$ is the multiplication map by r on M.
It is easy to see that μ_M is a homomorphism of R-algebras.
In general, μ_M is neither injective nor surjective.

Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let $D(-)$ be the Matlis dual functor $\mathrm{Hom}_R(-, E)$, where E is the injective hull of the field $R/\mathfrak{m}$

Definition:

Let R be a local ring. M has a canonical embedding

$$M \longrightarrow D(D(M)) = D^2(M),$$

$$m \longmapsto (\varphi \longmapsto \varphi(m))$$

into its bidual, this map will denoted by i_M. We will consider M as a submodule of $D^2(M)$ via i_M.

Definition. For an R-module M, **the cohomological dimension of M with respect to $\mathfrak{a}$** is defined as

$$\operatorname{cd}(\mathfrak{a}, M) := \max\{i \in \mathbb{Z} \mid H^i_{\mathfrak{a}}(M) \neq 0\}.$$

Introduction

Let $(R, \mathfrak{m})$ be a Noetherian local ring.

For a positive integer n, by using the theory of D-modules, Hellus showed that $H^n_{\mathfrak{a}}(D(H^n_{\mathfrak{a}}(R)))$ is either E or zero in the following cases:

(α) R **is a Noetherian local complete Cohen-Macaulay ring with coefficient field $R/\mathfrak{m}$ and there exists a regular sequence $x_1, \ldots, x_n \in \mathfrak{a}$ on R such that $\mathfrak{a} = (x_1, \ldots, x_n)$. In this case $\mathfrak{a}$ is a set-theoretic complete intersection ideal of R.**

(β) R **is a Noetherian local complete regular ring of equicharacteristic zero and a an ideal of height $n \geqslant 1$ such that there exists a regular sequence $x_1, \ldots, x_n \in \mathfrak{a}$ on R and $H^i_{\mathfrak{a}}(R) = 0$ for every $i > n$.**

Introduction

In [*], the present author obtained the following generalization of Hellus' Theorem.

Theorem Let R be a local ring and $\mathfrak{a}$ be an ideal of R such that $\mathfrak{a}M \neq M$ and $n := \operatorname{grade}_M \mathfrak{a} = \operatorname{cd}(\mathfrak{a}, M) \geqslant 1$. Then

$$H^n_{\mathfrak{a}}(D(H^n_{\mathfrak{a}}(M))) \cong D(M).$$

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By using this generalization in conjunction with spectral sequences method, Hellus and Stückrad, in [*], showed that:

if R is Noetherian local complete and $\mathfrak{a}$ an ideal of R such that $H^i_{\mathfrak{a}}(R) = 0$ for every $i \neq n(=$height $\mathfrak{a})$, then $\mu_{H^n_{\mathfrak{a}}(R)}$ is bijective.

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Moreover, Hellus and Stückrad, raised the following question:

If R is a commutative Noetherian complete local ring and $\underline{x} := x_1, \ldots, x_n$ is a regular sequence on R contained in $\mathfrak{a}$, when exactly is $J_{\underline{x},\mathfrak{a},R} := D(H^n_{\underline{x}R}(D(H^n_{\mathfrak{a}}(R))))$ zero?

where $\underline{x}R$ is the ideal $\sum_{i=1}^{n} x_i R$ of R.

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Let $(R, \mathfrak{m})$ be a Noetherian local ring and $\underline{\mathbf{x}} := x_1, \ldots, x_h$ a sequence of R. For every R-module M there is a canonical map

$$M/\underline{\mathbf{x}}M \xrightarrow{i_{M,\underline{\mathbf{x}}}} H^h_{\underline{\mathbf{x}}R}(M)$$

(coming from the description

$$H^h_{\underline{\mathbf{x}}R}(M) \cong \varinjlim_{n\in\mathbb{N}} M/(x_1^n, \ldots, x_h^n)M.)$$

Hellus and Stückrad

Theorem: **Let $(R,\mathfrak{m})$ be a Noetherian local complete ring and $\mathfrak{a}$ an ideal of R such that $H^{\ell}_{\mathfrak{a}}(R)=0$ for every $\ell \neq h = \mathrm{height}(\mathfrak{a})$. Set $H := H^{h}_{\mathfrak{a}}(R)$.Then**

1. $\mathrm{Hom}(\mathrm{H}, \mathrm{i}_{\mathrm{H}}) : \mathrm{End}(H) \longrightarrow \mathrm{Hom}(H, D^2(H))$ is an isomorphism.
2. There is a canonical isomorphism
$$\gamma_H : \mathrm{Hom}(H, D^2(H)) \longrightarrow D(H^h_{\mathfrak{a}}(D(H))).$$
3. $\mu_H : R \longrightarrow \mathrm{End}(H)$ is an isomorphism of R-algebras.
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The following conditions are equivalent:

1. $\sqrt{\mathfrak{a}} = \sqrt{(\underline{x}R)}$.
2. $\underline{x}$ is a sequence on D.
3. $D/\underline{x}D \xrightarrow{i_{D,\underline{x}}} H^h_{\underline{x}R}(D)$ is injective.
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Hellus and Stückrad

Question. Let $(R, \mathfrak{m})$ be a Noetherian local complete ring and $\mathfrak{a}$ an ideal of R, $h \in \mathbb{N}$; assume that $\underline{\mathbf{x}} := x_1, \ldots, x_h \in \mathfrak{a}$ is an R-regular sequence. When

$$J_{\underline{\mathbf{x}}, \mathfrak{a}, R} := 0.$$

Schenzel

Theorem. Let $(R, \mathfrak{m})$ be a Noetherian local Gorenstien ring of dimension n and $\mathfrak{a}$ an ideal of R such that dim$R/\mathfrak{a} = n - c$. Then there is a natural isomorphism

$$\mathrm{End}_R(H^c_{\mathfrak{a}}(R)) \cong \mathrm{Ext}^c_R(H^c_{\mathfrak{a}}(R), R)$$

Schenzel, Trung and Coung: 1978

Recall that we say a sequence of elements $x_1, \ldots, x_k$ of $\mathfrak{a}$ is an $\mathfrak{a}$-filter regular sequence on M if

$$x_i \notin \bigcup_{\mathfrak{p} \in \mathrm{Ass}_R\left(\frac{M}{(x_1, \ldots, x_{i-1})M}\right) \setminus V(\mathfrak{a})} \mathfrak{p}$$

for $i = 1, \ldots, k$.

Lemma. Let R is Noetherian, M is a finitely generated. If $x_1, \ldots, x_n$ be an $\mathfrak{a}$-filter regular sequence on M, then there is an element $x_{n+1} \in \mathfrak{a}$ such that $x_1, \ldots, x_n, x_{n+1}$ is an $\mathfrak{a}$-filter regular sequence on M.

Lemma. Let $n > 1$ and $x_1, \ldots, x_n$ be an $\mathfrak{a}$-filter regular sequence on M. Then

$$H^i_{\mathfrak{a}}(M) \cong \begin{cases} H^i_{(x_1,\ldots,x_n)}(M) & \text{for } 0 \leq i < n, \\ H^{i-n}_{\mathfrak{a}}(H^n_{(x_1,\ldots,x_n)}(M)) & \text{for } n \leq i. \end{cases}$$

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Proposition. For a non-negative integer n and an $\mathfrak{a}$-filter regular sequence $x_1, \dots, x_{n+1} \in \mathfrak{a}$ on M, there exists an exact sequence

$$0 \longrightarrow H^n_{\mathfrak{a}}(M) \longrightarrow H^n_{(x_1,\dots,x_n)}(M) \longrightarrow (H^n_{(x_1,\dots,x_n)}(M))_{x_{n+1}} \longrightarrow H^{n+1}_{(x_1,\dots,x_{n+1})}(M) \longrightarrow 0.$$

Proposition. Let n be a non-negative integer and $x_1, \ldots, x_n$ be an $\mathfrak{a}$-filter regular sequence on M. Let T be an $\mathfrak{a}$-torsion R-module. Then

$$\mathrm{Hom}_R(T, H^n_{\mathfrak{a}}(M)) \cong \mathrm{Hom}_R(T, H^n_{(x_1,\ldots,x_n)}(M)).$$

In particular

$$\mathrm{End}_R(H^n_{\mathfrak{a}}(M)) \cong \mathrm{Hom}_R(H^n_{\mathfrak{a}}(M), H^n_{(x_1,\ldots,x_n)}(M)).$$

Theorem. Let $\mathfrak{a}$ be a proper ideal of R and $n := \operatorname{grade}_R \mathfrak{a}$. Then, for every $\mathfrak{a}$-torsion R-module T, we have the following isomorphism

$$\operatorname{Hom}_R(T, H^n_{\mathfrak{a}}(R)) \cong \operatorname{Ext}^n_R(T, R).$$

In particular

$$\operatorname{End}_R(H^n_{\mathfrak{a}}(R)) \cong \operatorname{Ext}^n_R(H^n_{\mathfrak{a}}(R), R)$$

Theorem: **Let $\mathfrak{a}$ be a proper ideal of R such that $n := \mathrm{grade}_R \mathfrak{a} = \mathrm{cd}(\mathfrak{a}, R)$. Let $\mathrm{Ext}^i_R(R_z, R) = 0$ for all $i \in \mathbb{N}$ and $z \in \mathfrak{a}$. Then**

1. $\mathrm{End}_R(H^n_{\mathfrak{a}}(R))$ is a homomorphic image of R.
2. If moreover $\mathrm{Hom}_R(R_z, R) = 0$ for all $z \in \mathfrak{a}$, then $\mathrm{End}_R(H^n_{\mathfrak{a}}(R)) \cong R$ and so $\mu_{H^n_{\mathfrak{a}}(R)}$ is bijective.

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Corollary. Let $(R, \mathfrak{m})$ be a Noetherian local complete ring and $\mathfrak{a}$ an ideal of R such that $n := \mathrm{grade}_R\, \mathfrak{a} = \mathrm{cd}(\mathfrak{a}, R)$. Set $H := H^n_{\mathfrak{a}}(R)$. Then

$$\mu_H : R \longrightarrow \mathrm{End}_R(H)$$

is an isomorphism of R-algebras.

Khashyarmanesh and Khosh-Ahang

Theorem. **Let F be an R-linear covariant functor from $\mathcal{C}(R)$ to itself such that for every R-module L, $F(L)$ is $\mathfrak{a}$-torsion. Also let $c \in \mathbb{N}_0$ and $\mathfrak{a}$ be an ideal of R such that $\mathfrak{a}M \neq M$ and that $c \leqslant \operatorname{grade}(\mathfrak{a}, M)$. Then**

$$\mathcal{R}^0 F(H^c_{\mathfrak{a}}(M)) \cong \mathcal{R}^c F(M).$$

Theorem. **Let F be an R-linear covariant functor from $\mathcal{C}(R)$ to itself such that for every R-module L, $F(L)$ is $\mathfrak{a}$-torsion. Suppose that $\mathfrak{a}$ is an ideal of R and M is a finitely generated R-module such that $\mathfrak{a}M \neq M$ and that $c := \mathrm{cd}(\mathfrak{a}, M) = \mathrm{grade}(\mathfrak{a}, M)$. Then**

$$\mathcal{R}^i F(H^c_{\mathfrak{a}}(M)) \cong \mathcal{R}^{i+c} F(M)$$

for all $i \in \mathbb{N}_0$.

Theorem. Let M be a finitely generated R-module, $\mathfrak{a}$ be an ideal of R such that $\mathfrak{a}M \neq M$ and $c := \mathrm{cd}(\mathfrak{a}, M) = \mathrm{grade}(\mathfrak{a}, M)$. Then, for every ideal $\mathfrak{b}$ of R with $\mathfrak{b} \supseteq \mathfrak{a}$,

(i) $H^i_{\mathfrak{b}}(H^c_{\mathfrak{a}}(M)) \cong H^{i+c}_{\mathfrak{b}}(M)$, and;

(ii) $\mathrm{Ext}^i_R(R/\mathfrak{b}, H^c_{\mathfrak{a}}(M)) \cong \mathrm{Ext}^{i+c}_R(R/\mathfrak{b}, M)$

for all $i \in \mathbb{N}_0$.

Theorem. Let $(R, \mathfrak{m})$ be a Gorenstein local ring and $\mathfrak{a}$ be a cohomological complete intersection ideal of R. Set $c := \operatorname{cd}(\mathfrak{a}, R)$ and $d := \dim_R R/\mathfrak{a}$. Then

(i) $H^d_{\mathfrak{m}}(H^c_{\mathfrak{a}}(R)) \cong E(R/\mathfrak{m})$,

(ii) $\operatorname{Ext}^d_R(R/\mathfrak{m}, H^c_{\mathfrak{a}}(R)) \cong E(R/\mathfrak{m})$, and;

(iii) $H^i_{\mathfrak{m}}(H^c_{\mathfrak{a}}(R)) = 0 = \operatorname{Ext}^i_R(R/\mathfrak{m}, H^c_{\mathfrak{a}}(R))$ for all $i \neq d$.

Theorem. Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of an arbitrary commutative Noetherian ring R such that $\mathfrak{b} \supseteq \mathfrak{a}$, $\mathfrak{a}M \neq M$ and $c := \operatorname{grade}(\mathfrak{a}, M)$. Then

(i) we have a monomorphism from $H^c_{\mathfrak{b}}(M)$ to $H^c_{\mathfrak{a}}(M)$, and;

(ii) there exists a natural homomorphism from $\operatorname{End}(H^c_{\mathfrak{a}}(M))$ to $\operatorname{End}(H^c_{\mathfrak{b}}(M))$.

Sharp and Zakeri*

Module of generalized fractions

Let M be an R-module. The construction of a module of generalized fractions of M requires a (positive integer n and a) triangular subset $U \subseteq R^n$; the construction produces a module $U^{-n}M$, called the module of generalized fractions of M with respect to U, whose elements, called generalized fractions, have the form $\frac{m}{(u_1,\ldots,u_n)}$, where $m \in M$ and $(u_1, \ldots, u_n) \in U$.

[*] Sharp, R. Y. and Zakeri, H., Modules of generalized fractions, Mathematika 29 (1982), no. 1, 32–41.

O'Carroll*

The concept of a chain of triangular subsets on R is explained in [*]. Such a chain $\mathcal{U} = (U_i)_{i\in\mathbb{N}}$ determines a complex of modules of generalized fractions

$$0 \xrightarrow{d^{-1}} M \xrightarrow{d^0} U_1^{-1}M \longrightarrow \dots \longrightarrow U_i^{-i}M \xrightarrow{d^i} U_{i+1}^{-i-1}M \longrightarrow \dots,$$

in which $d^0(m) = m/(1)$ for all $m \in M$ and $d^i(m/(u_1, \dots, u_i)) = m/(u_1, \dots, u_i, 1)$ for all $i \in \mathbb{N}$, $m \in M$ and $(u_1, \dots, u_i) \in U_i$. We shall denote this complex by $C(\mathcal{U}, M)$.

[*] O'Carroll, L., On the generalized fractions of Sharp and Zakeri, J. London Math. Soc. (2) 28 (1983), no. 3, 417-427.

notations

Let $\underline{x} := x_1, \ldots, x_n$ be a sequence of elements of R. For each $i \in \mathbb{N}$, set

$$U(\underline{x})_i := \{(x_1^{\alpha_1}, \ldots, x_i^{\alpha_i}) \ : \ \text{there exists } j \text{ with } \ 0 \leqslant j \leqslant i \text{ such that } \alpha_1, \ldots, \alpha_j \in \mathbb{N} \text{ and } \alpha_{j+1} = \cdots = \alpha_i = 0\},$$

where x_r is interpreted as 1 whenever $r > n$. It is easy to see that, for each $i \in \mathbb{N}$, $U(\underline{x})_i$ is a triangular subset of R^i. We use $\mathcal{R}(\underline{x})$ to denote the family $(U(\underline{x})_i)_{i \in \mathbb{N}}$. Hence $\mathcal{R}(\underline{x})$ is a chain of triangular subsets on R. Write the associated complex $C(\mathcal{R}(\underline{x}), M)$ as

$$0 \xrightarrow{d^{-1}_{\underline{x},M}} M \xrightarrow{d^{0}_{\underline{x},M}} U(\underline{x})_1^{-1}M \longrightarrow \ldots \xrightarrow{d^{i}_{\underline{x},M}} U(\underline{x})_{i+1}^{-i-1}M \longrightarrow \ldots.$$

Proposition Let $\mathfrak{a}$ be a proper ideal of a Noetherian local ring R. Let $\underline{x} := x_1, \ldots, x_n (n > 0)$ be a regular sequence on M contained in $\mathfrak{a}$. Then there exists an exact sequence

$$0 \longrightarrow J_{\underline{x},\mathfrak{a},M} \longrightarrow D(D(M)) \longrightarrow D(H^{n-1}_{\underline{x}R}(D(\mathrm{Ker}\, d^{n}_{\underline{y},M})))$$

for every $x_{n+1} \in \mathfrak{a}$ such that $\underline{y} := x_1, \ldots, x_n, x_{n+1}$ is an $\mathfrak{a}$-filter regular sequence on M.

Theorem Let $(R, \mathfrak{m})$ be a Noetherian local ring and $\mathfrak{a}$ be a proper ideal of R. Let $\underline{x} := x_1, \ldots, x_n (n > 0)$ be a regular sequence on M in $\mathfrak{a}$. Suppose that there exists $x_{n+1} \in \mathfrak{a}$ such that $\underline{y} := x_1, \ldots, x_n, x_{n+1}$ is an $\mathfrak{a}$-filter regular sequence on M and $H^n_{\underline{x}R}(D(U(\underline{y})^{-n-1}_{n+1} M)) = 0$. Then

$$J_{\underline{x}, \mathfrak{a}, M} \cong D(D(M)).$$

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