We trust in him

The Role of the Syzygies of Local Cohomology Modules

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NOTATIONS AND TERMINOLOGY.

- R: Commutative Noetherian ring with non-zero identity
- F: A left exact covariant functor
- T: A right exact covariant functor
- M: An R-module

One of the problems which is recently in the center of considerations of some researchers, is investigating the behavior of left and right derived functors of local cohomology modules and their Matlis duals.
In this talk, we are going to investigate left and right derived functors of special local cohomology modules (or their Matlis duals) in terms of left and right derived functors of the R-module $\frac{R}{(\underline{x})}$ (which is simpler than the first one), where $\underline{x}$ is a standard system of parameters of R. In this way, we exploit of some relations between their syzygies.

DEFINITION.

A Serre subcategory $\mathcal{S}$ of the category of R-modules is a class of R-modules which is closed under taking submodules, quotients and extensions.

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- The class of weakly Laskerian modules is another example of Serre subcategory of the category of R-modules.

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- Recall that an R-module M is called *Matlis reflexive* if the canonical map $M \longrightarrow D_R(D_R(M))$ is an isomorphism, where $D_R(-)$ is the Matlis duality functor. By mapping a short exact sequence into its double dual and applying the snake lemma, one can deduce that the class of Matlis reflexive modules is also a Serre subcategory of the category of R-modules.

Assume that $(R, \mathfrak{m})$ is a commutative Noetherian local ring with dimension d. The Cohen-Macaulay defect of R, which is denoted by $\mathrm{cmd}R$, is defined as follows.

$$\mathrm{cmd}R = \dim_R R - \mathrm{depth}R$$

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Moreover, let M be a finitely generated R-module. A system of parameters $\underline{x} := x_1, \dots, x_n$ for M is said to be a standard system of parameters if

$$(\underline{x})H^i_{\mathfrak{m}}(M/(x_1, \dots, x_j)M) = 0$$

for all non-negative integers i and j with $i + j < d$.

THEOREM.

Let $(R, \mathfrak{m})$ be a d-dimensional local ring with $\operatorname{cmd} R \leq 1$ such that $H^i_{\mathfrak{m}}(R)$ is finitely generated for all $i < d$. Then for all positive integers k and n with $n \geq d$ and all standard systems of parameters $\underline{x}$ for R we have the following statements.

(i) If $R^{k-2}F(\frac{R}{(\underline{x})})$ and $R^{k+j}F(R)$ belong to $\mathcal{S}$ for all $j = -1, 0, 1, \ldots, d-1$, then $R^kF(H_{\mathfrak{m}}^{d-1}(R))$ also belongs to $\mathcal{S}$.

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(ii) If $R^{k-2}T(D_R(\frac{R}{(\underline{x})}))$ and $R^{k+j}T(E)$ belong to $\mathcal{S}$ for all $j=-1,0,1,\ldots,d-1$, then $R^kT(D_R(H_{\mathfrak{m}}^{d-1}(R)))$ also belongs to $\mathcal{S}$.

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(iii) If $L_{k-2}F(D_R(\frac{R}{(\underline{x})}))$ and $L_{k+j}F(E)$ belong to $\mathcal{S}$ for all $j=-1,0,\ldots,d-1$, then $L_nF(D_R(H^{d-1}_{\mathfrak{m}}(R)))$ also belongs to $\mathcal{S}$.

(iv) $R^nF(D_R(H_{\mathfrak{m}}^{d-1}(R))) \cong R^{n+2}F(D_R(\frac{R}{(\underline{x})}))$.

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(v) $R^nT(H_{\mathfrak{m}}^{d-1}(R)) \cong R^{n+2}T(\frac{R}{(\underline{x})})$.

(iv) $R^n F(D_R(H^{d-1}_{\mathfrak{m}}(R))) \cong R^{n+2} F(D_R(\frac{R}{(\underline{x})}))$.

(v) $R^n T(H^{d-1}_{\mathfrak{m}}(R)) \cong R^{n+2} T(\frac{R}{(\underline{x})})$.

(vi) $L_n F(H^{d-1}_{\mathfrak{m}}(R)) \cong L_{n+2} F(\frac{R}{(\underline{x})})$.

An ideal $\mathfrak{a}$ of R is called almost complete intersection if grade$(\mathfrak{a}, R) \geq \mu(\mathfrak{a}) - 1$, where $\mu(\mathfrak{a})$ is the number of elements of a minimal generating set of $\mathfrak{a}$.

A sequence $x_1, \ldots, x_n$ of elements of R is called a d-sequence on M if, for each $i = 0, 1, \ldots, n-1$, the equality

$$(\Sigma_{j=1}^{i} Rx_j)M :_M x_{i+1}x_k = (\Sigma_{j=1}^{i} Rx_j)M :_M x_k$$

holds for all $k \geq i+1$. Also, it is called an unconditioned strong d-sequence (u.s.d-sequence) on M if $x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}$ is d-sequence on M in any order for all $\alpha_1, \ldots, \alpha_n \in \mathbb{N}$.

DEFINITION.

Also, a sequence $x_1, \ldots, x_n$ of elements of an ideal $\mathfrak{a}$ of R is called an $\mathfrak{a}$-filter regular sequence on M if

$$\mathrm{Supp}_R\left(\frac{(x_1, \ldots, x_{i-1})M :_M x_i}{(x_1, \ldots, x_{i-1})M}\right) \subseteq V(\mathfrak{a})$$

for all $i = 1, \ldots, n$, where $V(\mathfrak{a})$ denotes the set of prime ideals of R containing $\mathfrak{a}$.
Now, we are ready to present one of our main results.

Let $\mathfrak{a}$ be an almost complete intersection ideal of a local ring R and r be the number of elements of a minimal generating set of $\mathfrak{a}$. Suppose that $H^i_{\mathfrak{a}}(R)$ is finitely generated for all $i < r$. Then there is a sequence $\underline{x}$ of elements of $\mathfrak{a}$ which is both an $\mathfrak{a}$-filter regular sequence and u.s.d-sequence on R such that the following statements hold.

(i) If $R^{k-2}F(\frac{R}{(\underline{x})})$ and $R^{k+j}F(R)$ belong to $\mathcal{S}$ for all $j = -1, 0, 1, \ldots, d-1$, then $R^kF(H^{r-1}_{\mathfrak{a}}(R))$ also belongs to $\mathcal{S}$.

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(ii) If $R^{k-2}T(D_R(\frac{R}{(\underline{x})}))$ and $R^{k+j}T(E)$ belong to $\mathcal{S}$ for all $j=-1,0,1,\ldots,d-1$, then $R^kT(D_R(H_{\mathfrak{a}}^{r-1}(R)))$ also belongs to $\mathcal{S}$.

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(iii) If $L_{k-2}F(D_R(\frac{R}{(\underline{x})}))$ and $L_{k+j}F(E)$ belong to $\mathcal{S}$ for all $j=-1,0,\ldots,d-1$, then $L_nF(D_R(H_{\mathfrak{a}}^{r-1}(R)))$ also belongs to $\mathcal{S}$.

(iv) $R^nF(D_R(H_{\mathfrak{a}}^{r-1}(R))) \cong R^{n+2}F(D_R(\frac{R}{(\underline{x})}))$.

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(v) $R^nT(H_{\mathfrak{a}}^{r-1}(R)) \cong R^{n+2}T(\frac{R}{(\underline{x})})$.

(vi) $L_nF(H_{\mathfrak{a}}^{r-1}(R))) \cong L_{n+2}F(\frac{R}{(\underline{x})})$.

By using the above two Theorems for well-known functors such as Hom and tensor, one can immediately gain some useful isomorphisms, which may be valuable in turn. We list them in the following.

COROLLARY.

Let $\mathfrak{a}$ be an almost complete intersection ideal of a local ring R and r be the number of elements of a minimal generating set of $\mathfrak{a}$. Suppose that $H^i_{\mathfrak{a}}(R)$ is finitely generated for all $i < r$. Then there is a sequence $\underline{x}$ of elements of $\mathfrak{a}$ which is both an $\mathfrak{a}$-filter regular sequence and u.s.d-sequence on R such that for all R-modules M and all $n \geq r$ we have the following isomorphisms.

(i) $Ext^n_R(H^{r-1}_{\mathfrak{a}}(R), M) \cong Ext^{n+2}_R(\frac{R}{(\underline{x})}, M)$

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(ii) $Ext_R^n(M, D_R(H_{\mathfrak{a}}^{r-1}(R))) \cong Ext_R^{n+2}(M, D_R(\frac{R}{(\underline{x})}))$

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(ii) $Ext^n_R(M, D_R(H^{r-1}_{\mathfrak{a}}(R))) \cong Ext^{n+2}_R(M, D_R(\frac{R}{(\underline{x})}))$

(iii) $Tor^R_n(M, H^{r-1}_{\mathfrak{a}}(R)) \cong Tor^R_{n+2}(M, \frac{R}{(\underline{x})})$

So, for all R-modules M with $\mathrm{injdim} M < n+2$, the R-module $Ext_R^n(H_{\mathfrak{a}}^{r-1}(R), M)$ is zero and for all R-modules M with $\mathrm{projdim} M < n+2$, we have

$$Ext_R^n(M, D_R(H_{\mathfrak{a}}^{r-1}(R))) = 0$$

and

$$Tor_n^R(M, H_{\mathfrak{a}}^{n-1}(R)) = 0.$$

Let $(R, \mathfrak{m})$ be a d-dimensional local ring with $\mathrm{cmd} R \leq 1$ such that $H^i_{\mathfrak{m}}(R)$ is finitely generated for all $i < d$. Then

(i) for all standard systems of parameters $\underline{x}$ for R, all R-modules M and all integers $k \geq d$ we have

$$Ext_R^k(H_{\mathfrak{m}}^{d-1}(R), M) \cong Ext_R^{k+2}(\frac{R}{(\underline{x})}, M);$$

$$Ext_R^k(M, D_R(H_{\mathfrak{m}}^{d-1}(R))) \cong Ext_R^{k+2}(M, D_R(\frac{R}{(\underline{x})})); \text{ and}$$

$$Tor_k^R(M, H_{\mathfrak{m}}^{d-1}(R)) \cong Tor_{k+2}^R(M, \frac{R}{(\underline{x})}).$$

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$$Tor_k^R(M, H_{\mathfrak{m}}^{d-1}(R)) \cong Tor_{k+2}^R(M, \frac{R}{(\underline{x})}).$$

(ii) for all R-modules M and all integers $k \geq d$, if there exists a standard system of parameters $\underline{x}$ for R such that $\operatorname{projdim}(\frac{R}{(\underline{x})}) < k+2$, then $Ext_R^k(H_{\mathfrak{m}}^{d-1}(R), M) = 0$.

(iii) for all R-modules M and all integers $k \geq d$, if there exists a standard system of parameters $\underline{x}$ for R such that $\operatorname{injdim}(D_R(\frac{R}{(\underline{x})})) < k+2$, then

$$Ext^k_R(M, D_R(H^{d-1}_{\mathfrak{m}}(R))) = 0.$$

(iii) for all R-modules M and all integers $k \geq d$, if there exists a standard system of parameters $\underline{x}$ for R such that $\operatorname{injdim}(D_R(\frac{R}{(\underline{x})})) < k+2$, then

$$Ext^k_R(M, D_R(H^{d-1}_{\mathfrak{m}}(R))) = 0.$$

(iv) for all R-modules M and all integers $k \geq d$, if there exists a standard system of parameters $\underline{x}$ for R such that $\operatorname{flatdim}(\frac{R}{(\underline{x})}) < k+2$, then $Tor^R_k(M, H^{d-1}_{\mathfrak{m}}(R)) = 0$.

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(vi) for all integers $k \geq d$ and all R-modules M with $\operatorname{projdim}(M) < k+2$, we have

$$Ext_R^k(M, D_R(H_{\mathfrak{m}}^{d-1}(R))) = 0$$

and

$$Tor_k^R(M, H_{\mathfrak{m}}^{d-1}(R)) = 0.$$

DEFINITION.

Here, we are going to investigate some properties of special local cohomology modules in $\mathcal{G}(R)$, where $\mathcal{G}(R)$ denotes the full subcategory of all Gorenstein R-modules. Recall that a finitely generated R-module M is called Gorenstein, if it satisfies in the following conditions:

(i) M is reflexive;

(ii) $Ext_R^i(M, R) = 0$ for all $i > 0$;

(iii) $Ext_R^i(M^*, R) = 0$ for all $i > 0$.

We end our talk by the following proposition.

PROPOSITION.

Suppose that $(R, \mathfrak{m})$ is a d-dimensional local ring with $\operatorname{cmd}(R) \leq 1$ such that $H^i_{\mathfrak{m}}(R)$ is finitely generated for all $i < d$. Then

(i) $H^{d-1}_{\mathfrak{m}}(R)$ is a non-free indecomposable module in $\mathcal{G}(R)$ if and only if there exists a standard system of parameters $\underline{x}$ such that $\frac{R}{(\underline{x})}$ is a non-free indecomposable module in $\mathcal{G}(R)$.

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(ii) if $H^{d-1}_{\mathfrak{m}}(R)$ is Gorenstein, then for all $i \geq d+1$ and all standard systems of parameters $\underline{x}$ of R, ith syzygy of $\frac{R}{(\underline{x})}$ is also Gorenstein.

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(iii) if there exists a standard system of parameters $\underline{x}$ of R such that $\frac{R}{(\underline{x})}$ is Gorenstein, then for all $i \geq n-1$, ith syzygy of $H^{d-1}_{\mathfrak{m}}(R)$ is also Gorenstein.

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