Prime Submodules and Spectral Spaces

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INTRODUCTION

- We establish conditions for the prime spectrum of an R-module M to be **Noetherian** and **spectral space**, with respect to the different topologies.

- Another main subject of this paper is presentation of conditions under which a module is **top**.

- We present some results about **minimal prime** submodules of certain modules.

Throughout this paper, R is a commutative ring with identity and all R-modules are unitary.

PRELIMINARIES

Let M be an R-module.

- A submodule N of an R-module M is said to be **prime** if $N \neq M$ and whenever $rm \in N$ (where $r \in R$ and $m \in M$), then $r \in (N :_R M)$ or $m \in N$ (see [Lu84]).

- The set of all prime submodules of M is called the **prime spectrum** of M and denoted by $\mathrm{Spec}(M)$. **Throughout this paper** X **denotes the prime spectrum** $\mathrm{Spec}(M)$ **of** M.

- Every maximal submodule of M is prime. The set of all maximal submodules of M is denoted by $\mathrm{Max}(M)$.

- For any submodule N of M we define

$$V(N) = \{P \in X \mid (P : M) \supseteq (N : M)\}$$

and

$$V^*(N) = \{P \in X \mid P \supseteq N\}.$$

Set

$$\mathsf{Z}(M) = \{V(N) \mid N \leq M\}$$

and

$$\mathsf{Z}^*(M) = \{V^*(N) \mid N \leq M\}.$$

Then the elements of the set $\mathrm{Z}(M)$ satisfy the axioms for closed sets in a topological space X (see [Lu99]). The resulting topology due to $\mathrm{Z}(M)$ is called the **Zariski topology relative to M** and denoted by τ.

There is another topology, τ^* say, on X due to $\mathrm{Z}^*(M)$ as the collection of all closed sets **if and only if** $\mathrm{Z}^*(M)$ is closed under finite union. When this is the case, we call the topology τ^* the **quasi-Zariski topology** on $\mathrm{Spec}(M)$ and M is called a **top** module (see [MMS97]).

- A topological space Y is said to be **Noetherian** if the open subsets of Y satisfy the ascending chain condition.
- A topological space Y is said to be **irreducible** if $Y \neq \emptyset$ and if every pair of non-empty open sets in Y intersect.
- Let Y be a closed subset of a topological space. An element $y \in Y$ is called a **generic point** of Y if $Y = Cl(\{y\})$.

- Following M. Hochster [Hoc69], we say that a topological space Y is a **spectral space** in the case where Y is homeomorphic to $\mathrm{Spec}(S)$, with the Zariski topology, for some ring S.

- Spectral spaces have been characterized by Hochster [Hoc69, Proposition 4] as the topological spaces Y which satisfy the following conditions:

  1. Y is a T_0-space[a];
  2. Y is quasi-compact[b];
  3. the quasi-compact open subsets of Y are closed under finite intersections and form an open base[c];
  4. each irreducible closed subset of Y has a generic point.

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[a] A topological space is T_0 if and only if the closures of distinct points are distinct.

[b] A topological space Y is *quasi-compact* if every collection of open subsets whose union is Y contains a finite subcollection whose union is Y.

[c] Let (Y, γ) be a topological space. Then A *base* for the topology γ is a collection B of subsets of Y such that $B \subseteq \gamma$ and for all $U \subseteq \gamma$, U is the union of some collection of sets taken from B.

MAIN RESULTS

Definition

Let M be an R-module. M is called **strongly top** if for every submodule N of M there exists an ideal I of R such that $V^*(N) = V^*(IM)$.

- Every multiplication[a] module is a strongly top module.
- Every strongly top R-module is a top module.
- It is not true that every top module is strongly top, for instance, the $\mathbb{Z}$-module $\mathbb{Q} \oplus \mathbb{Z}_p$, ($p$ is a prime integer), is a top module which is not strongly top.

[a] An R-module M is said to be a *multiplication* module (see [Bar81] and [EBS88]) if every submodule N of M is of the form IM for some ideal I of R.

Remark

- An R-module M is called **primeful** if either $M = (\mathbf{0})$ or $M \neq (\mathbf{0})$ and the map $\psi : \mathrm{Spec}(M) \to \mathrm{Spec}(\frac{R}{\mathrm{Ann}(M)})$ defined by $\psi(L) = (L : M)/\mathrm{Ann}(M)$ for every $L \in \mathrm{Spec}(M)$ be a surjective map (see [Lu07]). (e.g. finitely generated or faithfully flat modules.)

- Let M be an R-module. For every $x \in M$, we define $c(x) := \bigcap\{I | I$ is an ideal of R and $x \in IM\}$. A module M is called a **content** R-module if, for every $x \in M$, $x \in c(x)M$ (see [OR72]). (e.g. projective or faithful multiplication modules.)

- $\mathrm{rad}(\mathbf{0}) = \bigcap_{P \in \mathrm{Spec}(M)} P$.

Theorem

Suppose that M is an R-module.

1. Let M be strongly top. If either M is primeful or R is Noetherian, then (X, τ^*) is a **spectral space**. (This generalizes [ATOS10b, Theorem 4.9].)

2. Let R be a one-dimensional integral domain and let M be a content R-module such that $T(M) \subseteq \mathrm{rad}(\mathbf{0})$ and (X, τ) is a T_0-space. Then M is **top**. Moreover, if $\mathrm{Spec}(R)$ is Noetherian, then (X, τ^*) is **spectral**.[a]

3. If M is content and weak multiplication[b], then M is **top**. Moreover, if $\mathrm{Spec}(R)$ is Noetherian, then (X, τ^*) is **spectral**.

[a] Here $T(M)$ is the torsion submodule of M.

[b] an R-module M is called a *weak multiplication* if every prime submodule P of M is of the form IM for some ideal I of R (see [AS95] and [Azi03]).

Theorem

Suppose that M is an R-module.

1. Let R be a one-dimensional integral domain and let M be an R-module such that $T(M) \subseteq \mathrm{rad}(\mathbf{0})$. If (X, τ) is a T_0-space and the intersection of every infinite number of maximal submodules of M is zero, then M is **top** and (X, τ^*) is a **spectral space**.

   (This is a generalization of [ATOS10b, Theorem 4.11(d)].)

2. Let R be a one-dimensional integral domain with Noetherian spectrum and let M be a non-faithful top R-module. Then (X, τ^*) is a **spectral space**.

Theorem

Suppose that M is an R-module.

1. If M is distributive[a], then M is **top**.

2. If R is a one-dimensional integral domain with Noetherian spectrum, $T(M) \subseteq \mathrm{rad}(\mathbf{0})$ and (X, τ) is a T_0-space, then M is **top**.

3. If M is weak multiplication and R is a one-dimensional integral domain with Noetherian spectrum, then M is **top**. (This is a generalization of [ATOS10b, Theorem 3.18].)

[a] An R-module M is called *distributive* if the lattice of its submodules is distributive, i.e., $A \cap (B + C) = (A \cap B) + (A \cap C)$ and $A + (B \cap C) = (A + B) \cap (A + C)$ for all submodules A, B and C of M (see [Bar81]).

The next example shows that there is a $\mathbb{Z}$-module M such that (X, τ) is T_0 and $T(M) \subseteq \mathrm{rad}(\mathbf{0})$, but M is not weak multiplication.

Example

Consider the $\mathbb{Z}$-module $M = \mathbb{Z}(p^\infty) \oplus \mathbb{Z}$. For every prime ideal $\mathfrak{p} \in \mathrm{Spec}(\mathbb{Z})$ we have $|\mathrm{Spec}_{\mathfrak{p}}(M)| \leq 1$ and $T(M) = \mathrm{rad}(\mathbf{0})$. So, by the above Theorem, M is a top module. We note that M is **not** weak multiplication.

Corollary

The R-module M is **top** in each of the following cases:

1. R is a Dedekind domain and M is weak multiplication;
2. R is a one-dimensional integral domain with Noetherian spectrum and $\mathrm{Spec}(M) = \mathrm{Max}(M)$;
3. M is content and $\mathrm{Spec}(M) = \mathrm{Max}(M)$;

Corollary

Let M be an R-module. Then (X, τ^*) is a **spectral space** in each of the following cases:

1. R has Noetherian spectrum and M is multiplication;
2. M is content, $\mathrm{Spec}(M) = \mathrm{Max}(M)$ and R is Noetherian;

Proposition

Let M be a top R-module such that (X, τ^*) is a Noetherian space. Then M has only finitely many **minimal prime submodules**.

Corollary

In each of the following cases, the R-module M has only finitely many **minimal prime submodules**.

1. M is strongly top and R is Noetherian;
2. R is a one-dimensional integral domain with Noetherian spectrum and M is a content R-module such that $T(M) \subseteq \mathrm{rad}(\mathbf{0})$ and (X, τ) is a T_0-space;
3. M is content and weak multiplication and $\mathrm{Spec}(R)$ is Noetherian;
4. R is a one-dimensional integral domain and M is an R-module such that $T(M) \subseteq \mathrm{rad}(\mathbf{0})$ and (X, τ) is a T_0-space, and the intersection of every infinite number of maximal submodules of M is zero;

In the sequel, we present conditions under which (X, τ) is a spectral space.

Proposition

Let M be an R-module. Then (X, τ) is a **Noetherian** topological space in each of the following cases:

1. R satisfies ACC on radical ideals;
2. M satisfies ACC on radical submodules.

Remark

It is shown in [Lu10, Theorem 3.3], whenever M is a **primeful** R-module and $\mathrm{Spec}(R/\mathrm{Ann}(M))$ is a Noetherian topological space, then (X, τ) is a Noetherian topological space. We generalize this result in the next corollary.

Corollary

Let M be an R-module. Then (X, τ) is a **Noetherian** topological space in each of the following cases:

1. $\mathrm{Spec}(R)$ is a Noetherian topological space;
2. R is a Laskerian ring;
3. M is an Artinian R-module;
4. For every submodule N of M there exists a finitely generated submodule L of N such that $\mathrm{rad}(N) = \mathrm{rad}(L)$.

Remark

Some examples of non-Noetherian and non-primeful modules with Noetherian spectrum were introduced in [Lu10, Example 3.3].

For example, it is shown that the primeful $\mathbb{Z}$-module $M = \prod_{p\in\Omega} \mathbb{Z}/p\mathbb{Z}$, where Ω is the set of all prime integers p, the non-primeful and non-Noetherian $\mathbb{Z}$-modules $M = \bigoplus_{p\in\Omega} \mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Q}$ have Noetherian spectrum.

But, by the above Corollary, these $\mathbb{Z}$-modules have Noetherian spectrum. So, we can make plentiful examples of modules M such that (X, τ) is a Noetherian topological space without M being either primeful or Noetherian.

Remark

There are several examples of modules with Noetherian spectrum in [ATOS10a, Table of Examples 3.2] which the Noetherianness of its spectrum is **trivial** by the above Corollary.

Theorem

Let M be an R-module.

1. Assume that R is a ring with Noetherian spectrum and M is **flat**. Then (X, τ) is a **spectral space** if and only if it is a T_0-space.
2. If R is an integral domain with Noetherian spectrum and M is **torsion-free distributive**, then (X, τ) is a **spectral space**.
3. If R is a Dedekind domain and M is **torsion-free weak multiplication**, then (X, τ) is a **spectral space**.

Theorem

Let M be an R-module.

1. If R is a one-dimensional integral domain, M has at least one (0)-prime submodule and (X, τ) is a Noetherian space, then (X, τ) is a **spectral space** if and only if it is a T_0-space.

2. Let M be a **primeful** R -module. Then (X, τ) is a **spectral space** in each of the following cases:
   (a) M is strongly top;
   (b) M is distributive.

Example

Consider the non-torsion top $\mathbb{Z}$-module

$$M = \mathbb{Q} \oplus (\bigoplus_p \frac{\mathbb{Z}}{p\mathbb{Z}}).$$

M has one (0)-prime submodule, $T(M)$. The topological space (X, τ) is Noetherian (since $\mathbb{Z}$ is Noetherian) and T_0. Consequently, (X, τ) is a **spectral space** by the last theorem.

THANK YOU FOR YOUR ATTENTION

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