LOCAL HOMOLOGY AND GORENSTEIN FLAT MODULES

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1. Hyperhomology and Gorenstein flat modules.

Throughout this talk, R is a commutative Noetherian ring with nonzero identity.

We will work within $D(R)$, the derived category of R-modules.

The objects in $D(R)$ are complexes of R-modules and symbol $\simeq$ denotes isomorphisms in this category. For any complex X, its supremum and infimum are defined respectively by

$\sup X := \sup\{i \in \mathbb{Z} \mid H_i(X) \neq 0\}$ and $\inf X := \inf\{i \in \mathbb{Z} \mid H_i(X) \neq 0\}$.

$D_0(R)$: The full subcategory of complexes with homology modules concentrated in degree zero.

$D_{]}(R)$:The full subcategory of complexes that are homologically bounded to the right.

$D_{[}(R)$:The full subcategory of complexes that are homologically bounded to the left.

$D_{\square}(R)$:The full subcategory of homologically bounded complexes.

$D_{\square}^{f}(R)$:The full subcategory of homologically bounded complexes with finitely generated homology modules.

Modules will be considered as complexes concentrated in degree zero.

For any complex X in $D_{]}(R)$ (resp. $D_{[}(R)$), there is a bounded to the right (resp. left) complex P (resp. I) consisting of projective (resp. injective) R-modules which is isomorphic to X in $D(R)$.

A such complex P (resp. I) is called a projective (resp. injective) resolution of X. Also, for any complex $\boldsymbol{X}$ in $D_{]}(R)$, there is a bounded to the right complex F consisting of flat R-modules which is isomorphism to X in $D(R)$.

A complex X is said to have finite projective (resp. injective) dimension, if it possesses a bounded projective (resp. injective) resolution. Also , it is said to have finite flat dimension, if it possesses a bounded flat resolution.

We recall that the left derived tensor product functor $\sim \otimes_R^L -$ is computed by taking a projective resolution of the first argument or of the second one.

The right derived homomorphism functor $RHom_R(\sim, -)$ is computed by taking a projective resolution of the first argument or by taking an injective resolution of the second one. For any two convenient complexes X and Y and any integer i, set $Tor_i^R(X, Y) := H_i(X \otimes_R^L Y)$ and $Ext_R^i(X, Y) := H_{-i}(RHom_R(X, Y))$.

❖ Gorenstein flat dimension.

An R -module M is said to be Gorenstein flat if there exists an exact complex F of flat R-modules such that $M \cong im(F_0 \to F_{-1})$ and $J \otimes_R F$ is exact for all injective R -modules J .

Obviously, every flat R-module is Gorenstein flat.

The Gorenstein flat dimension of $X \in D_{]}(R)$, is defined by

$$Gfd_R X := \inf\{\sup\{l \in \mathbb{Z} \mid Q_l \neq 0\} \mid$$

Q is a bounded to the right complex of Gorenstein flat R-modules and $Q \simeq X\}$.

Note that for any complex $X \in D_{]}(R)$,

$$Gfd_R X \leq fd_R X$$

and equality holds if $fd_R X < \infty$.

❖ Dualizing complex.

A dualizing complex for R is a complex $D \in D^{f}_{\square}(R)$ such that the homothety morphism $R \to RHom_{R}(D, D)$, is an isomorphism in $D(R)$ and D has finite injective dimension.

Section 2:

Local homology

❖ Local homology

Let a be an ideal of R . The a-adic completion functor

$$\Lambda^{a}(-) = \varprojlim_{n} \left(\frac{R}{a^{n}} \otimes_{R} -\right)$$

defines an additive functor on the category of complexes of R -modules.

So, we may consider its left derived functor in the category $D(R)$.

For any complex $X \in D_{\sqsubset}(R)$, the complex $L\Lambda^{a}(X)$ is defined by

$L\Lambda^{a}(X) := \Lambda^{a}(F)$,where F is an (every) flat resolution of X .

Also, for any integer i, the i-th local homology module of X with respect

to a is defined by $H_{i}^{a}(X) := H_{i}(L\Lambda^{a}(X))$.

For any complex $X \in D_{]}(R)$, we have

$$width_R(a, X) = \inf L\Lambda^a(X),$$

where

$$width_R(a, X) := \inf(\frac{R}{a} \otimes^L_R X).$$

Let $\check{C}(\underline{a})$ denote the Cech complex on a set $\underline{a} = \{x_1, x_2, ..., x_n\}$

of generators of a . We have

$$L\Lambda^a(X) \simeq RHom_R(\check{C}(\underline{a}), X).$$

Section 3:

Local homology and Gorenstein flat modules

Lemma 3.1 Let $\mathfrak{a}$ be an ideal of R and $X \in D_{\sqsubset}(R)$. Then

$$\sup L\Lambda^{\mathfrak{a}}(X) \leq \sup(X) + cd_{\mathfrak{a}}(R),$$

Where $cd_{\mathfrak{a}}(R)$, denotes the supremum of i's such that i-th local cohomology module of R with respect to $\mathfrak{a}$ is nonzero.

The following result improves [M, Corllary 4.6].

Lemma 3.2 Let $\mathfrak{a}$ be an ideal of R. Then every Gorenstein flat R-module is $\Lambda^{\mathfrak{a}}$ -acyclic.

Lemma 3.3 Let a be an ideal of R and M a Gorenstein flat R-Module. Then there exists a natural isomorphism

$$\Lambda^{a}(M) \cong H_{0}^{a}(M).$$

Corollary 3.4 Let a be an R and

$$0 \to M' \to M \to M'' \to 0$$

is an exact sequence of Gorenstein flat R-modules. Then

$$0 \to \Lambda^{a} M' \to \Lambda^{a} M \to \Lambda^{a} M'' \to 0$$

is also exact.

Definition 3.5 To a morphism $\alpha : X \to Y$, the mapping cone complex of α is denoted by $\mu(\alpha)$ and is given by

$$\mu(\alpha)_l = Y_l \oplus X_{l-1}$$

and $\partial_l^{\mu(\alpha)}(y_l, x_{l-1}) = (\partial_l^Y(y_l) + \alpha_{l-1}(x_{l-1}), -\partial_{l-1}^X(x_{l-1}))$.

for every $l \in \mathbb{Z}$.

Lemma 3.6 Let $T : C_0(R) \to C_0(R)$ be a covariant additive functor. Any morphism $\alpha : X \to Y$ in $D(R)$ yields an isomorphism $\mu(T(\alpha)) \simeq T(\mu(\alpha))$ in $D(R)$.

Theorem 3.7 Let $\mathfrak{a}$ be an ideal of R.

If $X \in D_{]}(R)$ and Q is a bounded to the right complex of Gorenstein flat R-modules such that $X \simeq Q$, then

$$L\Lambda^{\mathfrak{a}}(X) \simeq \Lambda^{\mathfrak{a}}(Q)$$

and so

$$H_i^{\mathfrak{a}}(X) = H_i(\Lambda^{\mathfrak{a}}(Q))$$

for all $i \in \mathbb{Z}$.

Corollary 3.8 Let a be an ideal of R.

For any $X \in D_{]}(R)$, we have

$$\sup L\Lambda^{a}(X) \leq Gfd_{R} X \ .$$

The following result improves [FI,1.10].

Lemma 3.9 Let a be an ideal of R, $X \in D_{\square}(R)$ and $Y \in D^{f}_{]}(R)$.

Then

$$L\Lambda^{a}(X \otimes^{L}_{R} Y) \simeq L\Lambda^{a}(X) \otimes^{L}_{R} Y \ .$$

Lemma[FI,1.10]: Let a be an ideal of R, $X \in D_{\square}(R)$ and $Y \in D^{f}_{]}(R)$.

Such that $pd_R Y < \infty$. Then

$$L\Lambda^{a}(X \otimes_R^L Y) \simeq L\Lambda^{a}(X) \otimes_R^L Y .$$

Lemma 3.10 Let a be an ideal of R, $X \in D_{\square}(R)$ and $Y \in D^{f}_{]}(R)$.

Let Q be a bounded to the right complex of Gorenstein flat R-modules such that $Q \simeq X$ and F a flat resolution of Y . Then

$$L\Lambda^{a}(X \otimes^{L}_{R} Y) \simeq \Lambda^{a}(Q \otimes F).$$

[Ba1, 1.4.7] states that , $\Lambda^{a}(Q)$ is flat for all flat R-modules Q and all ideals a of R.

Question 3.11. Let a be an ideal of R and Q a Gorenstein flat R-module. Is $\Lambda^{a}(Q)$ Gorenstein flat ?

Definition 3.12 Large restricted flat dimension of an R-module M is defined by

$$Rfd_R(M) = \sup\{\sup(M \otimes_R^L T) \mid fd_R T < \infty\}.$$

Lemma 3.13 Let a be an ideal of R and Q a Gorenstein flat R-module. Then $Rfd_R \Lambda^a(Q) = 0$. Moreover, if R possesses a dualizing complex, then $\Lambda^a(Q)$ is Gorenstein flat.

The first part of the following corollary improves [CFH, Theorem 5.10 b)]

Corollary 3.14 Let R be a ring possessing a dualizing complex. Let

a be an ideal of R.

i) For any $X \in D_{]}(R)$, we have $Gfd_R L\Lambda^a(X) \leq Gfd_R(X)$.

ii) Let $Y \in D_{\square}^{f}(R)$, be a non-exact complex such that either its

projective or injective dimension is finite and $X \in D_{\square}(R)$.

Then

$$\sup L\Lambda^a(X \otimes_R^L Y) \leq Gfd_R X + \sup Y .$$

Theorem [CFH, Theorem 5.10 b)]

Let R be a ring possessing a dualizing complex. Let a be an ideal of R and $X \in D_{]}(R)$. If $Gfd_R(X) < \infty$, then $Gfd_R L\Lambda^{a}(X) < \infty$.

Proposition 3.15 Let R be a ring possessing a dualizing complex and $\mathfrak{a}$ an ideal of R. The following are equivalent:

i) $\Lambda^{\mathfrak{a}}(Q)$ is flat for all Gorenstein flat R-modules Q.

ii) $Gfd_R Q = fd_R Q$ for all $\mathfrak{a}$ -adic complete R-modules Q.

Next, we present a characterization of regularity of Gorenstein local rings.

Corollary 3.16 Let (R, m) be a local Gorenstein ring. The following are equivalent:

i) $\Lambda^{m}(Q)$ is flat for all Gorenstein flat R-modules Q.

ii) $Gfd_{R}Q = fd_{R}Q$ for all m -adic complete R-modules Q.

iii) R is regular.

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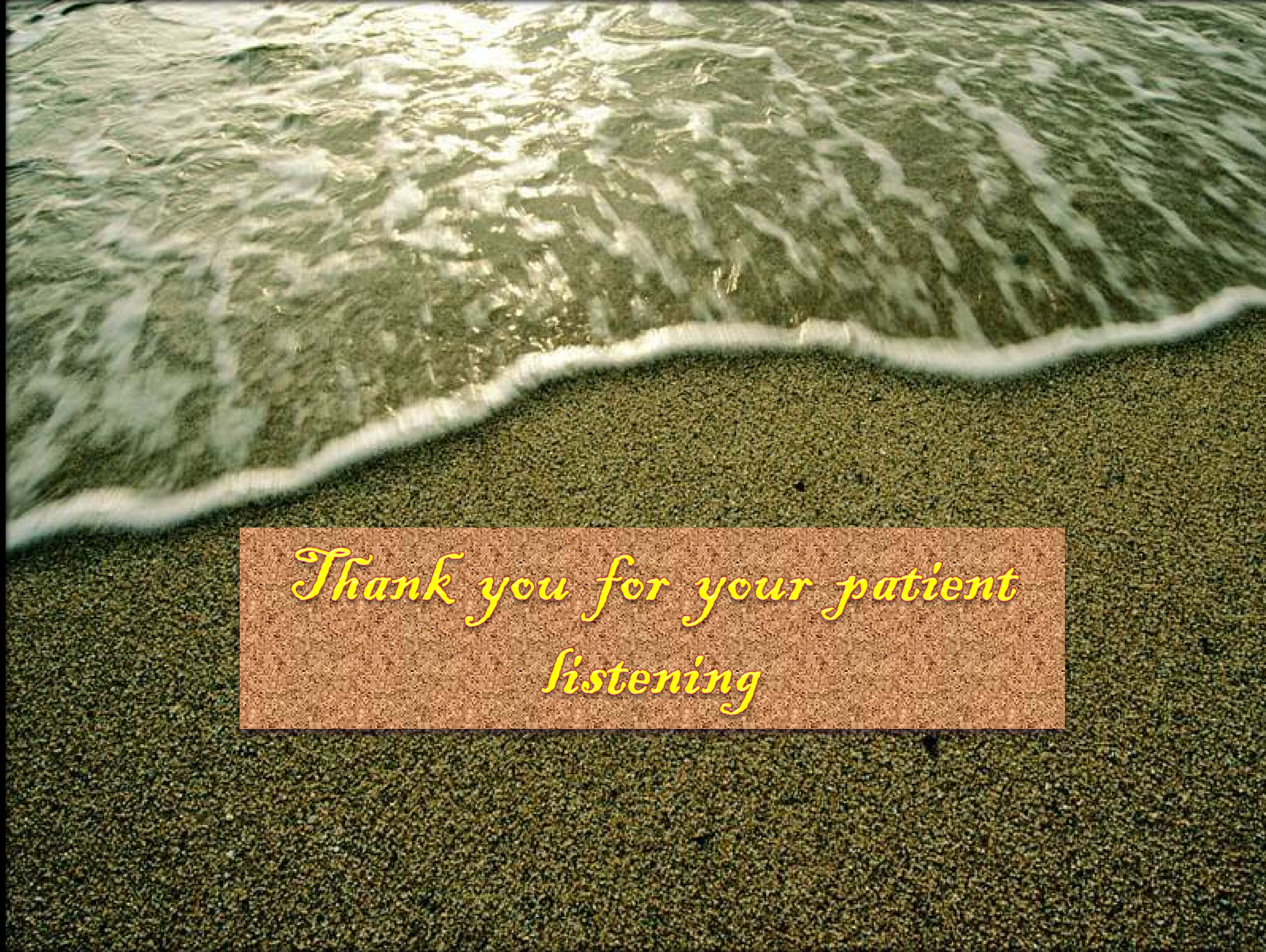
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